Einstein-Yang-Mills isolated horizons: Phase space, mechanics, hair, and conjectures

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(Received 7 March 2000; published 27 July 2000)

The concept of an ''isolated horizon'' has been recently used to provide a full Hamiltonian treatment of black holes. It has been applied successfully to the cases of *nonrotating*, *nondistorted* black holes in the Einstein vacuum, Einstein-Maxwell, and Einstein-Maxwell-dilaton theories. In this paper, the extent to which the framework can be generalized to the case of non-Abelian gauge theories is investigated in which ''hairy black holes'' are known to exist. It is found that this extension is indeed possible, despite the fact that, in general, there is no ''canonical normalization'' yielding a preferred horizon mass. In particular the zeroth and first laws are established for all normalizations. Colored static spherically symmetric black hole solutions to the Einstein-Yang-Mills equations are considered from this perspective. A canonical formula for the horizon mass of such black holes is found. This analysis is used to obtain nontrivial relations between the masses of the colored black holes and the regular solitonic solutions in Einstein-Yang-Mills theory. A general testing bed for the instability of hairy black holes in general nonlinear theories is suggested. As an example, the embedded Abelian magnetic solutions are considered. It is shown that, within this framework, the total energy is also positive and thus the solutions are potentially unstable. Finally, it is discussed which elements would be needed to place the isolated horizon framework for Einstein-Yang-Mills theory on the same footing as the previously analyzed cases. Motivated by these considerations and using the fact that the isolated horizon framework seems to be the appropriate language to state uniqueness and completeness conjectures for the EYM equations, in terms of the horizon charges, two such conjectures are put forward.

PACS number(s): 04.70.Bw, 04.20.Fy, 04.20.Jb

I. INTRODUCTION

The nonperturbative quantum geometry program, also known as ''loop quantum gravity,'' has met recently with substantial success in obtaining a calculation of the statistical mechanical entropy of a nonrotating black hole that accounts for its phenomenological identification with $\frac{1}{4}A$ [1]. In so doing it was necessary to introduce a complete classical Hamiltonian treatment of black holes (BH's). This was accomplished by generalizing and properly defining the sector of the theory that is going to be treated. This work was guided by the need to start with a well-defined action that would be differentiable in the sector under consideration. This led those authors to the specialization of the notion of the trapping horizons [2] of Hayward to that of ''isolated horizons.'' Physically the idea is to represent ''horizons in internal equilibrium and decoupled from what is outside.''

The zeroth and first laws of black hole mechanics refer to equilibrium situations and small departures therefrom. Therefore, in the standard treatments [3–8] one restricts oneself to stationary space-times admitting event horizons and perturbations off such space-times. The isolated horizon (IH) framework, which is tailored to more general physical situations, was introduced in [9] and the corresponding zeroth and first laws of black hole mechanics were established [10–12]. This framework generalizes the treatment of black hole mechanics in two directions. First, the notion of event horizons is replaced by that of ''isolated horizons,'' which can be defined quasilocally, unlike the former which can only be defined retroactively, after having access to the entire space-time history. Second, the underlying space-time need not admit any Killing field; isolated horizons need not be Killing horizons. The static event horizons normally used in black hole mechanics [3–5] are special cases of isolated horizons. Moreover, because one can now admit gravitational and matter radiation, there are many more examples. In particular, while the manifold $\mathcal{S}$ of static space-times admitting event horizons in the Einstein-Maxwell (EM) theory is, finite dimensional, the manifold $\mathcal{I}H$ of space-times admitting isolated horizons is *infinite* dimensional [11].

The resulting formalism is, then, not only a step in the construction of the quantum theory for the sector but also a new tool for studying classical aspects of black holes. When restricted to the static sector of the theory, it leads, for example, to an improvement in the treatment of the physical process version of the first law of black hole mechanics [10,11]. The formalism has so far only been applied to the Einstein-Maxwell system (with and without dilaton field), of which the known exact solutions [the Reissner Nordström (RN) solutions in EM and the so called Gibbons-Maeda solutions in Einstein-Maxwell-dilaton (EMD) theory] are particular examples.

In this work we will explore the extent to which this formalism can be extended to the Einstein-Yang-Mills theory where, in the static spherically symmetric sector, one finds the so-called colored black hole solutions [13–15]. The motivation is manyfold. First, we are interested in studying the robustness of the isolated horizons formalism in its ability to treat other theories, especially those where 'hairy' solutions are know to exist. From the isolated horizons perspective, this poses a special challenge, since now there is an apparent tension given by the mismatch between the number of con-

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served charges at infinity and at the horizon. In the colored black hole solutions, the only nonzero "charge" at infinity is the Arnowitt-Deser-Misner (ADM) mass, but the gauge field is nonvanishing at the horizon and it contributes to the "horizon magnetic charge." Is the isolated horizon framework robust enough to deal with this situation and to resolve this tension? Second, we would like to know whether the general formalism allows us to learn new facts about the static spherically symmetric (SSS) solutions. In particular, we want to understand what differences, if any, appear when we treat static but unstable black hole solutions, which is the case for the Abelian magnetic solutions and the higher n colored black holes. Finally, we want to investigate whether the formalism can—as can be expected given the fact that it yields a satisfactory treatment of the inner boundary in Hamiltonian terms—allow us to take the limit when the horizon area goes to zero and connect the black hole solutions with the regular, solitonic, solutions that are known to also exist in this theory.

We will in fact find answers to all of these questions and puzzles. We find, however, some subtleties that need to be stressed. In particular, the fact that there are no spherically symmetric solutions for some values of the horizon parameters poses a challenge to the isolated horizon framework, since there seems to be no canonical value for the horizon mass of the black hole in this situation. Thus, if the Einstein-Yang-Mills (EYM) system is to be in the same "status" as the Einstein-Maxwell system, some yet unknown part of the current scenario would have to yield to the tension mentioned above. The most natural resolution would entail the validity of a "uniqueness conjecture" for static solutions and a "completeness conjecture" for the existence of stationary solutions, which we will put forward. Their validity would guarantee the complete consistency of the formalism.

On the other hand, we will show some new results regarding SSS solutions as a nontrivial "application" of the formalism as it allows us to predict a relation between the ADM mass of a static black hole solution, its horizon mass, and the ADM mass of the corresponding solitonic solutions. These relations can be corroborated by numerical computations (consistent with the reported results in the literature). Therefore, these "coincidences" can be viewed, in a sense, as a check on the formalism.

This paper is organized as follows. In Sec. II, we recall some basic facts about the EYM system and about the known SSS solutions, both Abelian and non-Abelian. In Sec. III we specify the isolated horizon boundary conditions that we impose, making the necessary adjustments to incorporate non-Abelian gauge fields. Section IV deals with the action principle of the theory in the presence of the horizon as an inner boundary and the specification of the phase space of the theory. In Sec. V we consider the definition of surface gravity in the absence of a Killing field and we show the zeroth law of BH mechanics for general isolated horizons. The definition of horizon mass and the first law are studied in Sec. VI. Section VII is devoted to the completeness conjecture and some of its implication for stationary solutions. In Sec. VIII, we return to the study of static spherically symmetric solution to the EYM equations from the perspective of isolated black holes, and show some new results. Finally, we end with a discussion in Sec. IX.

Throughout the paper, we use units in which $c=G=1$ and the abstract index notation of Penrose (see Ref. [16]), except in Sec. VI where we use differential forms.

II. EINSTEIN-YANG-MILLS AND STATIC BLACK HOLES

This section has two parts. In the first one, we recall some basic facts about the Einstein-Yang-Mills system. In the second part, we briefly review the static spherically symmetric solutions to the EYM equations focusing on black hole solutions.

A. Einstein-Yang-Mills system

In the Einstein-Yang-Mills system, the gravitational part of the action, $S_{\rm grav}$ is given by

$$S_{\rm grav}=-\frac{1}{16\pi}\int_{\mathbf{M}}\sqrt{-g}R\,\mathrm{d}^4x, \tag{2.1}$$

and the matter part of the action is given by

$$S_{\rm YM}(\mathbf{A})=-\frac{1}{16\pi g_{\rm YM}^2}\int_{\mathbf{M}}\sqrt{-g}[\mathbf{F}^i_{ab}\mathbf{F}_i^{ab}]\mathrm{d}^4x, \tag{2.2}$$

where the abstract indices $a,b,\ldots$ denote space-time objects and the indices $i,j,\ldots$ are internal indices in the Lie algebra of the gauge group G. In this paper we shall consider $G=\mathrm{SU}(2)$. The field strength $\mathbf{F}_{ab}$ is given by $\mathbf{F}^i_{ab}=2\nabla_{[a}\mathbf{A}^i_{b]}+\epsilon^i{}_{jk}\mathbf{A}^j_a\mathbf{A}^k_b$, that is, the curvature of the Lie-algebra-valued one-form $\mathbf{A}^i_a$. The total action $S_{\rm tot}$ is given by

$$S_{\rm tot}=S_{\rm grav}+S_{\rm YM}. \tag{2.3}$$

We remind the reader that in the case of a non-Abelian Yang-Mills theory there is dimension-full parameter $g_{\rm YM}$ that, unlike the Abelian case, cannot be absorbed in the "gauge" fields. This endows the full theory with a natural scale at the classical level; i.e., $g_{\rm YM}^2$ has dimensions of mass.

The equations of motion that follow from $S_{\rm tot}$ are

$$D_a\mathbf{F}^{iab}=0, \tag{2.4}$$

$$R_{ab}=2\left(\mathbf{F}^i_{ac}\mathbf{F}^c_{ib}-\frac{1}{4}g_{ab}\mathbf{F}^2\right), \tag{2.5}$$

where $\mathbf{F}^2=\mathbf{F}^i_{ab}\mathbf{F}_i^{ab}$, and D_a is the generalized covariant derivative defined by $\mathbf{A}$. Furthermore, we have the Bianchi identity for the Yang-Mills sector:

$$D_{[c}\mathbf{F}^i_{ab]}=0. \tag{2.6}$$

The dual field tensor is given by ${}^*\mathbf{F}^i_{ab}=\frac{1}{2}\epsilon_{ab}^{cd}\mathbf{F}^i_{cd}$, where ϵ_{abcd} is the canonical volume element associated with g_{ab}.

Note that in contrast with the Einstein-Maxwell system, where, under appropriate falloff conditions, the conserved electric and magnetic charges can be defined at infinity, the naive expression,

$$\hat{Q}^i := \frac{1}{4\pi}\oint_{S_\infty} {}^*\mathbf{F}^i, \quad \hat{P}^i := \frac{1}{4\pi}\oint_{S_\infty} \mathbf{F}^i \tag{2.7}$$

fail to be gauge invariant. It is only in the presence of a globally defined isometry that the conserved charges might be invariantly defined (i.e., a natural gauge might be chosen). We can nevertheless define new gauge invariant quantities, for any two-sphere S as follows:

$$Q_S := \frac{1}{4\pi}\oint_S |{}^*\mathbf{F}|, \quad P_S := \frac{1}{4\pi}\oint_S |\mathbf{F}|, \tag{2.8}$$

where $|\mathbf{F}|_{ab}$ is the two-form defined in the following way: we take ϵ_{ab} the area two-form associated with the two-sphere S and define $f^i = \mathbf{F}^i_{ab}\epsilon^{ab}$. Then $|\mathbf{F}|_{ab} = \sqrt{\Sigma(f^i)^2}\epsilon_{ab}$. $|{}^*\mathbf{F}|_{ab}$ is analogously defined. In what follows, we shall refer to Q_S and P_S as the electric and magnetic charge contained ''within'' S, respectively.

B. Static solutions

Static spherically symmetric solutions to the EYM equations representing black hole space-times are known to exist in different situations (for a recent review see [17] and references therein).

A standard parametrization for the metric and gauge potential is given by

$$\mathrm{d}s^2 = -\left(1-\frac{2m(r)}{r}\right)e^{-2\delta(r)}\mathrm{d}t^2 + \left(1-\frac{2m(r)}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{2.9}$$

$$\mathbf{A} = a\tau_3\mathrm{d}t + b\tau_3\mathrm{d}r + (w\tau_1 + d\tau_2)\mathrm{d}\theta + (\cot\theta\tau_3 + w\tau_2 - d\tau_1)\sin\theta\mathrm{d}\phi, \tag{2.10}$$

where $a,b,w,$ and d are functions of (r,t).

One can then look for either static regular solutions by requiring $m(r) < r/2$ for all $r \geqslant 0$ [18] or for static black hole solutions with horizon at $r = r_H$ by requiring $m(r_H) = r_H/2$ and $m(r) < r/2$ for all $r \geqslant r_H$ [14,15]. Solutions of both types are found in the purely magnetic sector, for which, with a further gauge choice, one can set $a = b = d = 0$, and w becomes a function of r only.

The regular or solitonic solutions compose a discrete set parametrized by the number of nodes of the function $w(r)$ and are characterized by their ADM mass whose scale is set by g^2_{YM} [19].

On the other hand, for every value of r_H, one finds also two classes of black hole solutions: Abelian and non-Abelian.

1. Abelian solutions

The first class of solutions is given by what are essentially Abelian solutions embedded in SU(2). Within the Abelian sector we have either electrically charged or magnetically charged solutions. Unlike Einstein-Maxwell theory where the well-known Maxwell duality exists, in EYM theory there is no such duality and one is not allowed to treat them on ''equal footing.''

The electrically charged solutions with electric charge Q are given by the standard solutions that can be described by choosing $a \neq 0$ and $b = d = w = 0$ in Eq. (2.10). The YM potential is of the form $\mathbf{A} = (Q/r)\tau_3\mathrm{d}t$ and the metric is given by Eq. (2.9) with the functions $m(r) = M - (Q^2)/2r$ and $\delta = 0$. Note that these represent a two-parameter family of ''RN solutions'' with parameters M and Q.

The magnetically charged solutions with magnetic charge P are precisely the Reissner-Nordström solutions given by $w = 0$, $m(r) = M - (P^2)/2r$, and $\delta = 0$. Since we are considering the Abelian case, the magnetic charge P is not arbitrary but can take only one value $P = 1$. Note that the YM field strength, in the ''magnetic'' sector of the EYM theory takes the form

$$\mathbf{F} = w'\tau_1\mathrm{d}r\wedge\mathrm{d}\theta + w'\tau_2\sin\theta\mathrm{d}r\wedge\mathrm{d}\phi - (1-w^2)\tau_3\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\phi. \tag{2.11}$$

Thus, for the RN solutions where $w = 0$, we get from Eq. (2.11) that $P = 1$ for any sphere containing the black hole. The magnetically charged solutions are then parametrized by only one charge, namely, the ADM mass M.

These two sectors share the $Q = P = 0$ solution corresponding to the Schwarzschild solution. One can also construct dyonic solutions with both electric charge Q and unit magnetic charge. In all these solutions, one has to satisfy the inequality $r_H^2 \geqslant Q^2 + P^2$, in order to have black hole solutions with no naked singularities.

2. Non-Abelian colored black holes

These solutions correspond to the purely magnetic case, where for each value of the horizon area the equations have a discrete number of solutions which are strictly non-Abelian in nature (i.e., do not exit in the Abelian regime). These are labeled by an integer n that represents the number of nodes of the function $w(r)$. The lowest mode $n = 0$ represents the Schwarzschild solution. Therefore, the solution can be completely parametrized by two numbers (a_Δ, n), the horizon area and the integer n. All these solutions, for $n > 0$, are unstable under perturbations [36].

On the other hand, it is known that there are no nontrivial dyonic solutions (i.e., solutions with electric and magnetic fields) in the spherically symmetric sector [20].

From a historical perspective, these were the first examples of ''hairy black holes'' [13–15]. This is because the electric and magnetic charges (2.7) are both zero, so the only parameter at infinity is the ADM mass. If the no-hair conjecture were valid for the EYM system, the specification of M_{ADM} would suffice to characterize the solution completely. However, this is not the case, since for a given value of the ADM mass, there exist a countable number of *different* solutions, labeled by n. Equivalently we can label these solutions by the ADM mass M_{ADM} and n.

Even when the charges at infinity are not enough to specify static black holes uniquely, one might still hope to have quasilocal quantities defined at the horizon that are, in a sense, good coordinates for the manifold $\mathcal{S}$ of static solutions. Indeed, we shall put forward in the following sections a ''quasilocal uniqueness conjecture'' ($C1$): *All static BH solutions are characterized by its horizon parameters arising from the ''isolated horizon'' framework*. Let us refer to these quantities defined at the horizon as ''quasilocal parameters.'' In theories where no hair is present, as is the case of the Einstein-Maxwell-dilaton system, the number of ''quasilocal parameters'' equals the number of parameters at infinity labeling the static solutions [12]. Thus, stating a uniqueness conjecture in this theory is insensitive as to whether one is postulating it in terms of quantities at infinity (the standard viewpoint) or in terms of ''quasilocal charges.'' Our proposal is that, for general theories, one should state the postulate in terms of purely quasilocal quantities. In the EYM system, the quasilocal charges are a_Δ, the horizon area, and Q_Δ and P_Δ, the horizon electric and magnetic charges, respectively. In this case the first conjecture $C1$ reads: Given a triple of parameters $(a_\Delta, Q_\Delta, P_\Delta)$ for which a SSS solutions exists, then the solution is unique. However, note an apparent tension in this suggestion: Given the mismatch of the number of parameters at the horizon and at infinity, there is room for inconsistency when formulating, say, the laws of thermodynamics. This is because the number of ''independent'' parameters is different, when one considers charges at infinity or quasilocal parameters for, say, SSS colored solutions. As we shall see in Sec. VI, the nature of the problem can be made precise within the isolated horizon framework, and some ideas can be put forward to understand the origin of the difficulty.

For the convenience of the reader, in the next section we shall recall the notion of isolated horizons as defined in [9,11,12] and explore some of its consequences for EYM system of interest to this paper.

III. BOUNDARY CONDITIONS AND CONSEQUENCES

Let us recall the notion of isolated horizons Δ in general, and include in its definition the relevant modifications to incorporate the Einstein-Yang-Mills system. The basic boundary conditions defining Δ are the same as those introduced in [12].

Let us begin by recalling some notation. Fix any null surface $\mathcal{N}$, topologically $S^2\times R$, and consider foliations of $\mathcal{N}$ by families of spatial two-spheres. Given a foliation, we parametrize its leaves by $v=\mathrm{const}$ such that v increases to the future and set $n_a=-\nabla_a v$. Under a reparametrization $v\mapsto F(v)$, we have $n_a\mapsto F'(v)n_a$ with $F'(v)>0$. Thus, every foliation comes equipped with an equivalence class $[n_a]$ of normals n_a related by rescalings which are constant on each leaf.[1] Also recall that, given any one n_a, we can uniquely select a vector field l^a which is normal to $\mathcal{N}$ and satisfies $l^a n_a=-1$. (Thus, l^a is future pointing.) If we change the parametrization, l^a transforms via $l^a\mapsto[F'(v)]^{-1}l^a$. Thus, given a foliation, we acquire an equivalence class $[l^a,n_a]$ of pairs (l^a,n_a) of vector fields and one-forms on $\mathcal{N}$ subject to the relation $(l^a,n_a)\sim(G^{-1}l^a,Gn_a)$, where G is any positive function on $\mathcal{N}$ which is constant on each leaf of the foliation. Given a pair (l^a,n_a) in the equivalence class, we introduce a complex vector field m^a on $\mathcal{N}$, tangential to each leaf in the foliation, such that $m^a\overline{m}_a=1$. (By construction, $m^a l_a=m^a n_a=0$ on $\mathcal{N}$.) The vector field m^a is unique up to a phase factor. With this structure at hand, we now look at the main definition.

Definition: The internal boundary Δ of a space-time $(\mathbf{M},g_{ab})$ will be said to represent *a nonrotating isolated horizon* provided the following conditions hold.[2]

(i) *Manifold conditions:* Δ is a null surface, topologically $S^2\times R$.

(ii) *Dynamical conditions:* All field equations hold at Δ.

(iii) *Main conditions:* Δ admits a foliation such that the Newman-Penrose coefficients associated with the corresponding direction fields $[l^a,n_a]$ on Δ satisfy the following conditions:

(iiia) $\rho\triangleq-\overline{m}^a m^b\nabla_a l_b$, the expansion of $[l^a]$, vanishes on Δ.

(iiib) $\lambda\triangleq\overline{m}^a\overline{m}^b\nabla_a n_b$ and $\pi\triangleq l^a\overline{m}^b\nabla_a n_b$ vanish on Δ and the expansion $\mu:=m^a\overline{m}^b\nabla_a n_b$ of n_a is negative[3] and constant on each leaf of the foliation.

(iv) *Conditions on matter:* The Yang-Mills field $\mathbf{F}$ is such that

$$|\mathrm{Re}\,\phi_1|, \;\text{ and }\; |\mathrm{Im}\,\phi_1| \tag{3.1}$$

are constant on each leaf of the foliation introduced in condition (iii) [Recall that $\phi_1^i\triangleq\frac{1}{2}m^a\overline{m}^b(\mathbf{F}-i\,{}^\star\mathbf{F})^i_{ab}$], where $|\mathrm{Re}\,\phi_1|:=\sqrt{\Sigma_i(\mathrm{Re}\,\phi_1^i)(\mathrm{Re}\,\phi_1^i)}$, and $|\mathrm{Im}\,\phi_1|$ is analogously defined.

The first two conditions are quite tame: (i) simply asks that Δ be null and have appropriate topology while (ii) is completely analogous to the dynamical condition imposed at infinity. As the terminology suggests, (iiia) and (iiib) are the most important conditions. Note first that, if a pair (l^a,n_a) in the equivalence class $[l^a,n_a]$ associated with the foliation satisfies these conditions, so does any other pair $([G(v)]^{-1}l^a,G(V)n_a)$. Thus, the conditions are well defined. They are motivated by the following considerations. Condition (iiia) captures the idea that the horizon is isolated without having to refer to a Killing field. In particular, it

[1] These one-form fields n_a are defined intrinsically on $\mathcal{N}$. We can extend each n_a to the full space-time uniquely by demanding that the extended one-form be null. However, in this paper, we will not need this extension.

[2] Throughout this paper, the symbol $\triangleq$ will denote equality at points of Δ. For fields defined throughout space-time, an underarrow will denote pullback to Δ. The part of the Newman-Penrose framework [21] used in this paper is summarized in the Appendixes A and B of [11].

[3] For simplicity, in this paper we focus on black-hole-type horizons rather than cosmological ones. To incorporate interesting cosmological horizons, one has to weaken this condition and allow the possibility that μ is everywhere positive on Δ. See [11].

implies that the area of each two-sphere leaf in the foliation is the same. We will denote this area by a_Δ and define the *horizon radius* r_Δ via $a_\Delta = 4\pi r_\Delta^2$.

Condition (iiib) has three sets of implications. First, one can show that if, as required, one can find a foliation of Δ satisfying (iiib), that foliation is *unique*. (In the SSS family, as one might expect, this condition selects the foliation to which the rotational Killing fields are tangential.) Second, it implies that the imaginary part of (the Newman-Penrose Weyl component) Ψ_2, which captures angular momentum, vanishes and thus restricts us to *nonrotating* horizons. Third, the requirement that the expansion μ of n^a be negative implies that Δ is a *future* horizon rather than *past* horizon [2]. Finally, consider the spherical symmetry requirement on the Yang-Mills field component ϕ_1^i given by condition (iv). While this condition is a strong restriction, it can be motivated by analogy with the Einstein-Maxwell case. (For further motivation and remarks on these conditions, see [9,11].)

Since these conditions are *local* to Δ, the notion of an isolated horizon is quasilocal; in particular, one does not need an entire space-time history to locate an isolated horizon. Furthermore, the boundary conditions allow for the presence of radiation in the exterior region; thus, space-times admitting isolated horizons need not admit any Killing field [22]. Indeed, the manifold $\mathcal{IH}$ of solutions to field equations admitting isolated horizons is infinite dimensional [11].

In spite of this generality, boundary conditions place strong restrictions on the structure of various fields *at* Δ. Let us begin with conditions on the Yang-Mills field. The stress-energy tensor T_{ab} of $\mathbf{F}$ satisfies the dominant energy condition. Hence, on Δ, $-T_{ab}l^b$ is a future-directed, causal vector field. Now, using the Raychaudhuri equation and field equations *at* Δ [condition (ii) of the main definition], we conclude $T_{ab}l^a l^b \triangleq 0$. By expanding out this expression [see Eq (2.5)] we obtain

$$\mathbf{F}^i_{ab} \triangleq \phi_1^i 2(l_{[a}n_{b]} - m_{[a}\overline{m}_{b]}) + \phi_2^i 2(m_{[a}l_{b]}) + \text{c.c.}, \tag{3.2}$$

for *some* complex-algebra-valued functions ϕ_1^i and ϕ_2^i (the only nonvanishing Newman-Penrose components of $\mathbf{F}^i_{ab}$) on Δ, where c.c. stands for the "complex conjugate term." These equations say that there is no flux of Yang-Mills radiation across Δ. Finally, condition (iv) in the main definition implies

$$|\mathrm{Re}\,\phi_1| \triangleq \frac{2\pi}{a_\Delta} Q_\Delta, \quad |\mathrm{Im}\,\phi_1| \triangleq \frac{2\pi}{a_\Delta} P_\Delta, \tag{3.3}$$

where Q_Δ is the electric charge and P_Δ the magnetic charge at the horizon as defined by Eq. (2.8). Thus the boundary conditions severely restrict the form of matter fields at Δ. The component $\phi_0^i = -l^a m^b \mathbf{F}^i_{ab}$ of the YM field vanishes and the gauge-invariant components of ϕ_1^i are completely determined by the electric and magnetic charges. However, the component ϕ_2^i of the YM field is unconstrained.

Restrictions imposed on space-time curvature at Δ are essentially the same as in Ref [11].[4] Results relevant to this paper can be summarized as follows. In the Newman-Penrose notation, for the Ricci tensor components, we have

$$\Phi_{00} = \frac{1}{2}R_{ab}l^a l^b \triangleq 0, \quad \Phi_{01} = \frac{1}{2}R_{ab}l^a m^b \triangleq 0,$$

$$\Phi_{11} = \frac{1}{4}R_{ab}(l^a n^b + m^a \overline{m}^b) \triangleq 8\pi^2 \frac{(Q_\Delta^2 + P_\Delta^2)}{a_\Delta^2},$$

$$R \triangleq 0, \tag{3.4}$$

where R is the scalar curvature. The Weyl tensor components satisfy

$$\Psi_0 = C_{abcd}l^a m^b l^c m^d \triangleq 0, \quad \Psi_1 = C_{abcd}l^a m^b l^c n^d \triangleq 0,$$

$$\Psi_2 = C_{abcd}l^a m^b \overline{m}^c n^d \triangleq \Phi_{11} - \frac{2\pi}{a_\Delta}. \tag{3.5}$$

Furthermore,

$$\Psi_3 \triangleq \Phi_{21}, \quad \text{that is,} \quad C_{abcd}l^a n^b \overline{m}^c n^d \triangleq \frac{1}{2}R_{ab}\overline{m}^a n^b. \tag{3.6}$$

As expected, for the SSS solutions discussed in Sec. II, these conditions are satisfied. We note that even when the curvature components Ψ_2 and Φ_{11} are the only ones different from zero in SSS solutions, for a general isolated horizon other curvature components (such as Ψ_3 and Ψ_4) may be "dynamical," i.e., vary along the integral curves of l.

In the Einstein-Maxwell-dilaton system [12], we have a set of parameters $(a_\Delta, Q_\Delta, P_\Delta, \phi_\Delta)$ at the horizon (and the same number at infinity), and those parameters were naturally selected as the horizon parameters. In the EYM case we have seen that we can define electric and magnetic charges at the horizon (Q_Δ, P_Δ). The boundary conditions ensure that these quantities are constant along the horizon Δ (and explicitly gauge invariant). Thus, it is natural to use the triplet $a_\Delta, Q_\Delta, P_\Delta$ to parametrize general EYM isolated horizons. Let us denote by $\mathcal{L}$ the space of horizon parameters with coordinates $(a_\Delta, Q_\Delta, P_\Delta)$, with the following restrictions: $a_\Delta > 4\pi(Q_\Delta^2 + P_\Delta^2)$ and $Q_\Delta, P_\Delta \in [0,\infty)$.

IV. ACTION AND PHASE SPACE

The gravitational action has been shown to be differentiable, with respect to variations respecting the isolated horizon boundary conditions, in [9,11]. We refer the reader to those papers for details. One important property of the varia-

[4]This is because these restrictions were obtained assuming rather general conditions on the matter stress-energy which are satisfied in EYM theory. The derivation of some of these results involves long calculations and a topological result on the Chern class of the SO(2) connection associated with the dyad $(m, \overline{m})$. See [11].

tional principle is that one is varying, in the pure gravitational case, histories with a fixed value a_Δ^0 of the horizon area. We also need to ensure that the matter action is differentiable and work out the Hamiltonian framework for the matter sector as well. This could require imposition of additional boundary conditions on matter fields, but as we will see, the conditions already imposed on matter, described in Sec. III, will be enough. The purpose of this section is concentrate on the variational principle for the Yang-Mills field and work out the Hamiltonian description.

Since we require that field equations hold on Δ, the gravitational boundary conditions already imply certain restrictions on the behavior of YM fields there. As noted in Sec. III, boundary conditions imply that several components of the Ricci tensor vanish on Δ, and that the curvature tensor $\mathbf{F}$ has the form (3.2). In particular $|\mathrm{Re}(\phi_1)|$ and $|\mathrm{Im}(\phi_1)|$ are spherically symmetric on the preferred cross sections. In that case, $|\boldsymbol{\phi}_1|$ can be expressed in terms of the electric and magnetic charges Q_Δ and P_Δ of the isolated horizon:

$$Q:\triangleq-\frac{1}{4\pi}\oint_{S_v}|{}^*\mathbf{F}|\triangleq\frac{1}{2\pi}\oint_{S}|\mathrm{Re}\,\boldsymbol{\phi}_1|^2\boldsymbol{\epsilon}, \tag{4.1}$$

$$P:\triangleq-\frac{1}{4\pi}\oint_{S_v}|\mathbf{F}|\triangleq\frac{1}{2\pi}\oint_{S_v}|\mathrm{Im}\,\boldsymbol{\phi}_1|^2\boldsymbol{\epsilon}. \tag{4.2}$$

[Here S_v are the two-spheres $v=\mathrm{const}$ in the preferred foliation. The minus signs in front of the first integrals in Eqs. (4.1) and (4.2) arise because we have oriented S_v such that the radial normal is in going rather than outgoing.]

Let us now discuss the action principle. For the same reasons that the area a_Δ is kept fixed in the variational principle, we will now restrict ourselves to histories for which the values of electric and magnetic charges on the horizon are fixed to Q_Δ^0 and P_Δ^0, respectively. To make the action principle well defined, we need to impose suitable boundary conditions on the YM fields. Conditions at infinity are the standard ones given in [23]. To find boundary conditions on Δ, let us consider the standard YM bulk action

$$S_{\mathrm{YM}}=-\frac{1}{16\pi}\int_{\mathbf{M}}\sqrt{-g}\,\mathbf{F}^i_{ab}\mathbf{F}_i^{ab}\mathrm{d}^4x. \tag{4.3}$$

Variation of S_{YM} yields

$$\delta(S_{\mathrm{YM}})=-\frac{1}{4\pi}\int_{\mathbf{M}}(D_a\mathbf{F}^{abi})\,\delta\mathbf{A}_i\sqrt{-g}\,\mathrm{d}^4x+\frac{1}{4\pi}\int_{\partial\mathbf{M}}\delta\mathbf{A}^i_{[a}{}^*\mathbf{F}^i_{bc]}\tilde{\eta}^{abc}\mathrm{d}^3x. \tag{4.4}$$

As usual, the bulk term provides the equations of motion provided the surface term vanishes. The boundary term (4.4) at infinity is the usual one and is dealt with in the standard manner [23]. When evaluated at the horizon, the boundary term (4.4) does not automatically vanish. On Δ, the boundary term in Eq. (4.4) can be written as

$$\frac{3}{4\pi}\int_{\Delta}\delta\mathbf{A}^i_a{}^*\mathbf{F}_{ibc}l^{[c}\tilde{\eta}^{ab]}dv\,d^2x. \tag{4.5}$$

Now, Eq. (3.2) implies that on Δ the pullback of $\mathbf{F}^i_{ab}l^b$ vanishes. Then, the horizon contribution reduces to

$$\int\ \mathrm{d}v\ \oint_{S_v}\delta(\mathbf{A}_{ia}l^a)\,{}^*\mathbf{F}^i_{ab}\tilde{\eta}^{ab}\mathrm{d}^2x, \tag{4.6}$$

where, as before, v is the affine parameter (with respect to the horizon metric) along the integral curves of l^a such that $v=\mathrm{const}$ defines the preferred foliation of Δ and S_v are the two-spheres in this foliation. Now, since isolated horizons are to be thought of as "nondynamical," it is natural to ask that the gauge field $\mathbf{A}^i_a$ be invariant under the action of l. This is equivalent to asking that $\mathcal{L}_l\underleftarrow{\mathbf{A}}^i_a\triangleq\underleftarrow{D}_a\mathcal{V}^i_l$ be satisfied [24] for $\mathcal{V}^i_l$ the gauge generator. This condition is naturally satisfied since the pullback of $\mathbf{F}^i_{ab}l^b$ to the horizon Δ vanishes. Furthermore, the form of the boundary term (4.6) suggests that we fix the gauge so that $(\mathbf{A}^i\cdot l):=A^i_al^a$ is proportional to $\mathrm{Re}(\phi^i_1)$, that is, $(\mathbf{A}^i\cdot l)=c\,\mathrm{Re}(\phi^i_1)$ for c a constant on our space of histories.[5] The norm of $(\mathbf{A}^i\cdot l)$, as we shall see in following sections, is determined by the consistency of the Hamiltonian formalism (and will take its standard value in the static solutions, when they exist). Then the boundary term (4.6) arising in the variation of the action vanishes; i.e., the bulk action itself is differentiable and the action principle is well-defined. Note that the permissible gauge transformations are now restricted: If $\mathbf{A}_a\mapsto\mathbf{A}_a+D_af$, the generating "function" f^i has to satisfy $l^aD_af^i\triangleq0$ on Δ (as well as satisfy standard falloff conditions at infinity).

Let us now provide a summary of the structure of the phase space of Yang-Mills fields. Fix a foliation of $\mathbf{M}$ by a family of spacelike three-surfaces M_t (level surfaces of a time function t) which intersect Δ in the preferred two-spheres. Fix a normalization for the vector field l^a. Denote by t^a a "time-evolution" vector field which is not tangential to the foliation with affine parameter t which tends to a unit time translation at infinity and to the vector field l^a on Δ. The canonical conjugate moment is given by

$$\tilde{\mathbf{\Pi}}^a_i=\frac{\sqrt{h}}{4\pi}h^{ac}\mathbf{n}^b\mathbf{F}_{ibc}, \tag{4.7}$$

where h_{ab} is the intrinsic metric on M_t induced from g_{ab}, and $\mathbf{n}^a$ is the normal to M_t. In terms of the lapse and shift fields N and N^a defined by t^a, the Legendre transform of the action yields

[5]This gauge condition was independently found by Ashtekar, Fairhurst, and Krishnan in the more general context of *distorted* horizons [25]. We thank S. Fairhurst for communicating their results prior to publication.

$$S_{\rm EM}=\frac{1}{4\pi}\int dt\int_{M_t}d^3x\left(\tilde{\mathbf{\Pi}}^a_i\nabla_a(\mathbf{A}^i\cdot l)-N^dF^i_{ad}\tilde{\mathbf{\Pi}}^{ia}+\frac{N}{2\sqrt{h}}h_{ab}\tilde{\mathbf{\Pi}}^{ia}\tilde{\mathbf{\Pi}}^b_i+\frac{N}{2}\sqrt{h}F^{ab}_iF^i_{ab}\right), \tag{4.8}$$

where the two-form $\mathbf{E}_{iab}$ is the pullback to M of $^*\mathbf{F}_{iab}$, $^{(*)}\mathbf{E}^i_a:=\frac{1}{2}\epsilon_a{}^{bc}\mathbf{E}^i_{bc}$, and $^{(*)}\mathbf{F}^i_a:=\frac{1}{2}\epsilon_a{}^{bc}\mathbf{F}^i_{bc}$, with ϵ_{abc} the volume form defined by h_{ab}. The two-form $\mathbf{E}^i_{ab}$ is related to the canonically conjugate moment as follows: $\mathbf{E}^i_{ab}=(1/4\pi)\underset{\sim}{\eta}_{abc}\tilde{\mathbf{\Pi}}^{ic}$.

Thus, as usual, the phase space consists of pairs $(\mathbf{A}^i_a,\mathbf{E}_{abi})$ on the three-manifold M_t, subject to boundary conditions, where the connection $\mathbf{A}^i_a$ is now the pullback to M_t of the Yang-Mills four-potential and the two-form $\mathbf{E}^i_{ab}$ is the dual of the electric field vector density. These fields are subject to boundary conditions. On any horizon two-sphere S_v, condition (iv) must hold, ensuring that the pullbacks of $|\mathbf{F}|_{ab}$ and $|\mathbf{E}|_{ab}$ are spherically symmetric.[6] [Since $(\mathbf{A}^i\cdot l)$ appears as a Legendre multiplier in Eq. (4.8), it is not part of the phase space variables at Δ.] At infinity, $\mathbf{A}^i_a,\mathbf{E}_{iab}$ are subject to the appropriate boundary conditions that ensure asymptotic flatness of the metric and are general enough to include the nontrivial EYM solutions mentioned in Sec. III. The conditions for the YM sector are [23]

$$A^i_a=A^i_a(\theta,\phi)/r+o(r^{-1}),\quad F^i_{ab}=F^i_{ab}(\theta,\phi)/r^2+o(r^{-2}). \tag{4.9}$$

The symplectic structure on this YM sector of phase space can be read off from Eq. (4.8):

$$\Omega|_{(\mathbf{A},\mathbf{E})}(\delta_1,\delta_2)=\frac{1}{4\pi}\int_M[\delta_1\tilde{\mathbf{E}}^a_i\delta_2\mathbf{A}^i_a-\delta_2\tilde{\mathbf{E}}^a_i\delta_1\mathbf{A}^i_a]. \tag{4.10}$$

(The asymptotic conditions ensure that the integrals converge.) As usual, there is one first class constraint $D_{[a}\mathbf{E}^i_{bc]}=0$ which generates gauge transformations: Under the canonical transformation generated by $\int f_iD_{[a}\mathbf{E}^i_{bc]}\tilde{\eta}^{abc}d^3x$, the canonical fields transform, as usual, via $\mathbf{A}^i_a\mapsto\mathbf{A}^i_a+D_af^i$ and $\mathbf{E}_{iab}$ ‘‘rotates.’’ Note that our boundary conditions allow the generating function f_i to be nontrivial on the (intersection of M_t with) Δ; the smeared constraint function is still differentiable. Thus, as in the gravitational case, the gauge degrees of freedom do *not* become physical in this framework.

Let us summarize. By imposing a set of boundary conditions on the horizon, to be satisfied by *all* histories in the variational principle, we arrived at the phase space of EYM isolated horizons $\mathcal{IH}_0$. This phase space can be seen as the (gauge equivalence class of) solutions to the equations of motion with constant and fixed quasilocal parameters a^0_Δ, Q^0_Δ, and P^0_Δ. For the purpose of the variational principle (in the sense that is well defined and yields the correct equation of motion), it is enough to consider configurations with fixed values of the horizon parameters. However, as we shall see in the following sections, when considering the Hamiltonian formulation—essential for the formulation of the first law—one needs to extend the space of isolated horizons from $\mathcal{IH}_0$ to $\mathcal{IH}$, where *all* possible values of the quasilocal parameters are considered [11].

[6]At the horizon Δ, the natural parameter v and the ‘‘time’’ parameter t coincide.

V. SURFACE GRAVITY AND THE ZEROTH LAW

In each SSS solution there is a unique time-translational Killing field t^a which is unity at infinity. As usual, the surface gravity $\kappa_{\rm SSS}$ is defined in terms of its acceleration at the horizon: $t^a\nabla_at^b\triangleq\kappa_{\rm SSS}t^b$. Unlike the Einstein-Maxwell-dilaton theory where knowledge of the exact solutions allows us to write κ in terms of the parameters of the solutions, for the YM field we do not have a closed form for κ. The only expression we have at our disposal is the general formula found by Visser for SSS space-times of the form (2.9), independently of the matter content of the theory, given by the expression [26]

$$\kappa_{\rm SSS}=\frac{1}{2r_{\rm h}}e^{-\delta(r_{\rm h})}[1-2m'(r_{\rm h})]. \tag{5.1}$$

From the perspective of the isolated horizon framework, κ is the acceleration of the properly normalized null normal l^a to Δ [10,11]. In the SSS solutions, Δ happens to be a Killing horizon and we can select a unique vector field l^a from the the equivalence class $[l^a]$ simply by setting $l^a\triangleq t^a$. Then $\kappa_{\rm SSS}$ is the acceleration of this specific l^a. In the case of general isolated horizons, the challenge is to find a prescription to single out a preferred l^a, without reference to any Killing field. The strategy we would like to adopt has two steps and is motivated by the one used for isolated horizons in dilaton gravity [12]. However, as we shall see below, we will encounter some difficulties which make the EYM case more subtle than the systems previously studied.

In the first step, we will normalize l^a *only* up to a constant, leaving a rescaling freedom $l^a\mapsto l'^a=cl^a$, where c is a constant on Δ but may depend on the parameters $r_\Delta,Q_\Delta,P_\Delta$ of the isolated horizon. For each such l^a, we can define the surface gravity κ_l *relative to that* l^a via $l^a\nabla_al^b\triangleq\kappa_ll^b$. Rescaling of l^a now induces a ‘‘gauge transformation’’ in κ: $\kappa_l\mapsto\kappa_{l'}=c\kappa_l$. (Recall that in the general Newman-Penrose framework, κ is a connection component and therefore undergoes the standard gauge transformations under a change of the null tetrad. By fixing l^a up to a constant rescaling, we have reduced the general gauge freedom to that of a constant rescaling.) Since the zeroth law only says that the surface gravity is constant on Δ, if it holds for one l^a, it holds for every $l'^a=cl^a$. Thus, for the zeroth law, it is in fact *not* essential to get rid of the rescaling freedom.

Recall that the isolated horizon is naturally equipped with equivalence classes $[l,n]$ of vector and covector fields, subject to the relation $(l,n)\sim(G^{-1}l,Gn)$ for any positive function $G\equiv G(v)$ on Δ. As we already mentioned, our first task is to reduce the freedom in the choice of $G(v)$ to that of a constant. We use the same strategy as in [10,11]. (For motivation, see [11].) Recall that μ, the expansion of n, is strictly negative and constant on each leaf of the preferred foliation: $\mu \equiv \mu(v) < 0$. It is easy to verify that

$$n^a \mapsto G(v) n^a \quad \text{implies} \quad \mu \mapsto G(v) \mu(v). \tag{5.2}$$

Hence, we can *always* use the $G(v)$ freedom to set $\mu \triangleq$ const. This condition restricts the family of (l^a, n_a) pairs and reduces the equivalence relation to $(l^a, n_a) \sim (c l^a, c^{-1} n_a)$ where c is any constant on Δ. We will denote the restricted equivalence class by $[l^a, n_a]_R$. In the second step, we can *arbitrarily fix* the numerical value of μ in terms of the parameters of the isolated horizon and eliminate the rescaling freedom altogether, thereby selecting, *for each choice* of μ, a canonical pair (l^a, n_a) on each isolated horizon.

With the equivalence class $[l^a, n_a]_R$ at our disposal, as discussed above, we can define a surface gravity κ_l via $l^a \nabla_a l^b \triangleq \kappa_l l^b$. Constancy of κ_l on Δ follows from the same arguments that were used in [11]. For completeness, let us briefly recall the structure of that proof. First, using conditions on derivatives of l^a, n_a introduced in the main definition, one can express the self-dual part of the Riemann curvature in terms of $\kappa_l, \nabla_a \kappa_l, \mu$ (and another field which is not relevant to this discussion). Comparing this expression to the standard Newman-Penrose expansion of the self-dual curvature tensor in terms of curvature scalars [21], and using the fact that certain curvature scalars vanish on Δ [see Eqs. (3.4) and (3.5)], one can conclude

$$(\nabla_{[a} \kappa_l) n_{b]} \triangleq 0 \quad \text{and} \quad \kappa_l \triangleq \frac{\Psi_2}{\mu}. \tag{5.3}$$

The first equation implies that κ_l is spherically symmetric. Hence, it only remains to show that $\mathcal{L}_l \kappa_l \triangleq 0$. Since μ is now a constant on Δ, it suffices to show that $\mathcal{L}_l \Psi_2 = 0$. Now, the (second) Bianchi identity implies that

$$\mathcal{L}_l (\Psi_2 - \Phi_{11}) \triangleq 0. \tag{5.4}$$

Finally, using Eq. (3.2), we conclude that $\Phi_{11} = 8\pi^2 (Q_\Delta^2 + P_\Delta^2)/a_\Delta^2$. Thus, Φ_{11} is constant on Δ. Combining these results, we conclude $\mathcal{L}_l \kappa_l \triangleq 0$, whence κ_l is constant on Δ. This establishes the zeroth law.

Let us now consider the second step in fixing the normalization of l. So far, we have only required that μ be a (negative) constant but not fixed its value. Under the rescaling $\mu \mapsto c^{-1} \mu$ we have $l \mapsto c l$ and $\kappa_l \mapsto c \kappa_l$. Hence, the remaining rescaling freedom in l and κ can be exhausted simply by fixing the value of μ in terms of the isolated horizon parameters. We would like to single out a canonical choice, and the obvious strategy is to fix μ to the value μ_{SSS} it takes on the SSS solutions. However, there are two difficulties: First, although μ_{SSS} is a well-defined function of the isolated horizon parameters r_Δ, Q_Δ and P_Δ, where static solutions exist, there is no closed expression for it in terms of these parameters simply because the full set of solutions (including the colored black holes) is not known in closed form. The second and more serious problem is that, for an arbitrary point in parameter space, there might not be any SSS solution (recall that the colored black holes span only a countable number of points in the P_Δ axis for a given value of r_Δ). As we shall see in next section, one can still have a consistent Hamiltonian formulation and a first law even for those isolated horizons lying on points of the parameter space where no SSS solutions exist. But as should be clear from the discussion, at this point there is no canonical normalization of μ on the whole space $\mathcal{IH}$.

Even when we cannot find an explicit functional form for μ, we can still write the general form that κ_l shall have. Using an identity coming from the isolated horizons boundary condition we have the following relation [11]:

$$\kappa_l \triangleq \frac{1}{\mu}\left(-\frac{2\pi}{a_\Delta} + \Phi_{11}\right). \tag{5.5}$$

Using the expression for Φ_{11} in terms of the charges (3.4) we have

$$\kappa_l = -\frac{1}{\mu}\frac{1}{2r_\Delta^2}\left[1 - \frac{(Q_\Delta^2 + P_\Delta^2)}{r_\Delta^2}\right]. \tag{5.6}$$

For those points of parameter space $\mathcal{L}$ where an SSS static solution exists, we can go further and make use of the general form of the metric (2.9). One then finds that the expansion of the properly normalized n is such that

$$\mu_{\text{SSS}} = -\frac{e^{\delta(r_\Delta) - \delta(\infty)}}{r_\Delta}. \tag{5.7}$$

A coordinate transformation can always render $\delta(\infty) = 0$, so the properly normalized κ for general isolated horizons takes the form

$$\kappa = \frac{e^{-\delta(r_\Delta, Q_\Delta, P_\Delta)}}{2r_\Delta}\left[1 - \frac{(Q_\Delta^2 + P_\Delta^2)}{r_\Delta^2}\right]. \tag{5.8}$$

It is in a sense remarkable that our ignorance about the explicit form of the solutions is encoded in the function δ.

We end this section with a discussion. As we have emphasized in this section, any choice of μ in terms of the horizon parameters defines a vector field l^a and a surface gravity on $\mathcal{IH}$. However, one would like to make contact with the space of static solutions in such a way that the surface gravity κ_l coincides with the surface gravity of the properly normalized Killing field. In the case of Einstein-Maxwell-dilaton theory [12] this was indeed possible since the static solutions span the space of horizon parameters $\mathcal{L}$. That is, for each value of the isolated horizon parameters, there exists a (unique) static solution which allows us to fix μ in a unique, canonical way. In the case of EYM theory, the space of SSS solutions does not span $\mathcal{L}$. Thus, one is able to canonically fix κ only in this subspace; for a general point, there is no preferred normalization. This seems to be a serious shortcoming of the formalism that might place the EYM system on a different status than the EM and EMD systems. As we shall see in next section, this ambiguity is also mani-

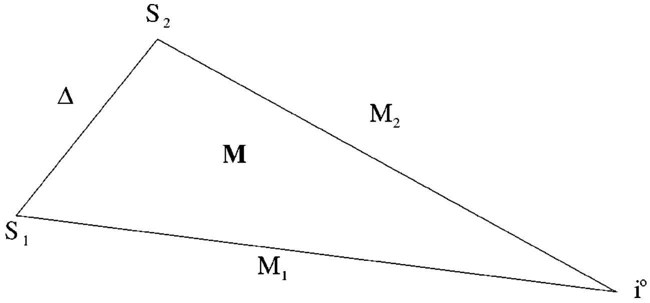

FIG. 1. Region **M** of space-time considered in the variational principle is bounded by two partial Cauchy surfaces M_1 and M_2. They intersect the isolated horizon Δ in preferred two-spheres S_1 and S_2 and extend to spatial infinity i^0.

fested in the definition of mass. However, this problem will also motivate a second conjecture that we shall put forward in Sec. VII.

VI. MASS AND THE FIRST LAW

The space-times that admit an isolated horizon are not necessarily stationary; therefore, it is no longer meaningful to identify the ADM mass M with the mass M_Δ of the isolated horizon. For the formulation of the first law, we must first introduce an appropriate definition of M_Δ. This definition should be general enough in the sense that, for each choice for the normalization of l^a, the mass should be uniquely defined, and the first law should be valid for *any* choice of normalization. As we shall see, the Hamiltonian framework provides a natural strategy. In the Einstein-Maxwell-dilaton case the total Hamiltonian consists of a bulk term and *two* surface terms, one at infinity and the other at the isolated horizon. As usual, the bulk term is a linear combination of constraints and the surface term at infinity yields the ADM energy. In a rest frame adapted to the horizon it is then natural to identify the surface term at Δ as the horizon mass M_Δ. Indeed, there are several considerations that support this identification [11].

For the gravitational part of the action and Hamiltonian, the discussion of [11] only assumed that the stress-energy tensor satisfies two conditions at Δ: (i) $-T_{ab}l^a$ is a future-pointing causal vector field on Δ, and (ii) $T_{ab}l^a n^b$ is spherically symmetric on Δ. Both these conditions are met in the present case. Therefore, we can take over the results of [11] directly. For the matter part of the action and Hamiltonian, the overall situation is again analogous, although there are the obvious differences in the detailed expressions. As in the Einstein-Maxwell case, matter terms contribute to the surface terms in the Hamiltonian only because one has to perform one integration by parts to obtain the Gauss constraint in the bulk term.

The net result is the following. Consider a foliation of the given space-time region **M** by a one-parameter family of (partial) Cauchy surfaces M_t, each of which extends from the isolated horizon Δ to spatial infinity i^0 (see Fig. 1). We will assume that M_t intersects Δ in a two-sphere belonging to our preferred foliation and that the initial data induced on M_t are asymptotically flat. Denote by S_Δ and S_∞ the two-sphere boundary of M_t at the horizon and infinity, respectively. Choose a timelike vector field t^a in **M** which tends to the unit time-translation orthogonal to the foliation at spatial infinity and to the vector field l^a on Δ, with the normalization fixed (or partially fixed) as in Sec. V. Then, the Hamiltonian H_t generating evolution along t^a is given by

$$H_t=\int_{M_t}\text{constraints}+\lim_{r_0\to\infty}\oint_{S_{r_0}}\left(\frac{r_0}{4\pi G}\Psi_2\right)^2\epsilon-\Phi_\infty Q_\infty + \oint_{S_\Delta}\left(\frac{\mu^{-1}}{4\pi G}\Psi_2\right)^2\epsilon+|(\mathbf{A}\cdot l)|Q_\Delta+V, \tag{6.1}$$

where S_{r_0} are large two-spheres of radius r_0 and V is a constant on $\mathcal{IH}_0$, the space of isolated horizons with fixed values of the horizon parameters. (The calculation and the final result are completely analogous to those in the Einstein-Maxwell case [11].) Note that the surface terms depend only on the ''Coulombic'' parts of the gravitational and Yang-Mills fields.

It is easy to check that the surface term at infinity is, as usual, the time component $P_a^{\mathrm{ADM}}t^a$ of the ADM four-momentum P_a^{ADM}, which in the present $(-,+,+,+)$ signature is the negative of the ADM energy, $P_a^{\mathrm{ADM}}t^a=-E^{\mathrm{ADM}}$. It is natural to identify the surface term at S_Δ as the energy of the isolated horizon. (There is no minus sign because S_Δ is the *inner* boundary of M.) Since $t^a \triangleq l^a$ and since l^a represents the ''rest frame'' of the isolated horizon, this energy can in turn be identified with the horizon mass M_Δ. Thus, we have

$$M_\Delta^{(l)}=\oint_{S_\Delta}\left(\frac{\mu^{-1}}{4\pi G}\Psi_2\right)^2\epsilon+|(\mathbf{A}\cdot l)|Q_\Delta+V_{(l)}(a_\Delta,Q_\Delta,P_\Delta). \tag{6.2}$$

Here, the so far undetermined function V depends only on the horizon parameters (and coupling constants). In the variational principle, this term played no role, but in the Hamiltonian description it becomes essential, since we are now interested in variations along the full isolated horizons phase space $\mathcal{IH}$. Thus, one should be able to consider in the formalism displacements along directions in which the horizon parameters change. As we shall show below, requiring a consistent Hamiltonian formulation enables us to determine the function V for the EYM system. Now, using the expression (5.3) of surface gravity in terms of the Weyl tensor (and μ), and calling $\Phi_\Delta := |\mathbf{A}\cdot l|$ on Δ, we can cast M_Δ in a more familiar form

$$M_\Delta=\frac{1}{4\pi}\kappa a_\Delta+\Phi_\Delta Q_\Delta+V(a_\Delta,Q_\Delta,P_\Delta), \tag{6.3}$$

where we have dropped the explicit l^a dependence of the mass for notational simplicity. Thus, as in the Einstein-Maxwell case, we obtain a Smarr formula. However, the meaning of various symbols in the equation is somewhat different. Since an isolated horizon need not be a Killing horizon, in general M_Δ does not equal the ADM mass; nor is

κ or Φ_Δ computed using a Killing field. Since the constraints are satisfied in any solution, the bulk term in Eq. (6.1) vanishes as well. Hence, in this case, $H_t = M_\Delta - E^{\rm ADM}$, the difference being the ''radiative energy'' in the space-time. Finally, as emphasized in [11], the matter contribution to the mass formula (6.2) is subtle: while it does not include the energy in radiation outside the horizon, it does include the energy in the ''Coulombic part'' of the field associated with the black hole hair. (Recall that the future limit of the Bondi energy has this property.) This fact is crucial to the analysis of the ''physical process version'' of the first law. However, since this issue was discussed in detail in [11], we shall not discuss this here.

Now, a consistent Hamiltonian formulation (for a sector of a diffeomorphism invariant theory) requires that for an arbitrary vector δ tangent to the symplectic manifold (i.e., the phase space Γ), one has

$$\delta H = \Omega(\delta, X_H), \tag{6.4}$$

where X_H is the vector field that corresponds to the equations of motion for a given choice of lapse and shift, and H is the Hamiltonian function corresponding to the same choice of lapse and shift. In the preceding prescription one assumes that the evaluation of δH is carried out by considering the change in H associated with the displacement δ of the phase space point, but keeping the lapse and shift *fixed*. If we now let the choice of lapse and shift depend on the phase space point—as is the case when l is normalized as in Sec. V—we would obtain a *new* variation $\delta\tilde{H}$. This might fail to satisfy

$$\delta\tilde{H} = \Omega(\delta, X_H), \tag{6.5}$$

with the *same* X_H as in Eq. (6.4). It turns out that the necessary and sufficient condition to obtain the required consistency is the validity of the first law [27]:

$$\delta M_\Delta = \frac{1}{8\pi}\kappa\,\delta a_\Delta + \Phi_\Delta \delta Q_\Delta . \tag{6.6}$$

That is, the first law of black hole mechanics—for quantities defined only at the horizon—arises naturally as part of the requirements for a consistent Hamiltonian formulation in which, for every value of the horizon parameters, one has chosen a canonical lapse and shift functions making the latter dependent on the point; in phase space. Note that in contrast with the above situation, when constructing a Hamiltonian to deal with the analogous problem at infinity, the canonical choice of normalization of lapse and shift at infinity is taken as independent of the phase space point; namely, they are chosen to correspond to a unit time translation (normal to the initial-data hypersurface) at infinity.

In the cases of Einstein vacuum, Einstein-Maxwell, and Einstein-Maxwell-dilaton theories, this consistency requirement translates into an identity that is automatically satisfied by the expressions of M_Δ, κ, etc., for all values of the parameters and variations thereof. In the case of Einstein-Yang-Mills theory—as well as in other theories where hair is present—the only way to ensure the validity of the consistency requirement is to limit the class of variations δ allowed. This has two dramatic consequences: First, it defines a foliation of phase space by a collection of (symplectic) leaves over which the Hamiltonian formulation is consistent, a situation which puts the construction in the ''nonstandard'' class. In fact, some simple systems are described by a similar type of situation, as, for example, the (reduced) Hamiltonian description of a rotating body where the—three-dimensional—phase space is foliated by two spheres, each of which is a true symplectic manifold where the Hamiltonian motion is restricted [28,29]. Second, the status of the first law changes from that of an identity, valid for all variations, to that of a specification of the class of variations that the formalism allows.

It is important to note that in the first law (6.6) only variations of the electric charge are involved, and not variations of the magnetic charge. On the other hand, the horizon mass (6.3) might depend on P_Δ through V.

Let us now see that asking consistency of the formalism leads us to some conditions that the function V should satisfy. In order to do this we shall follow, for completeness, Ref. [30] closely. The first step in this direction is to regard Eq. (6.6) as an identity between one forms $\mathrm{d}M_\Delta = (1/8\pi)\kappa \mathrm{d}a_\Delta + \Phi_\Delta \mathrm{d}Q_\Delta$, where κ, Φ_Δ, and V are functions on $\mathcal{L}$. Thus one can consider differential forms on $\mathcal{L}$ and take an exterior derivative of the ''first law'' to arrive at

$$0 = \frac{1}{8\pi}\mathrm{d}\kappa \wedge \mathrm{d}a_\Delta + \mathrm{d}\Phi_\Delta \wedge \mathrm{d}Q_\Delta . \tag{6.7}$$

The first conclusion coming from Eq. (6.7) is that the variations on $\mathcal{L}$ are restricted to submanifolds such that the pullback of the form $\mathrm{d}P_\Delta \wedge \mathrm{d}a_\Delta$ vanishes. That is, P_Δ is not free to vary independently of r_Δ and Q_Δ. This is precisely what happens for SSS solutions [representing only a *discrete* set of curves in the plane (r_Δ, P_Δ)] and in the static axial-symmetric case [31] (also covering a discrete set of curves in the plane). From now on, we restrict ourselves to the symplectic leaves where the formalism is well defined. On these submanifolds the magnetic charge becomes a function of the area and electric charge, $P_\Delta = P_\Delta(r_\Delta, Q_\Delta)$.

Second, the condition (6.7) gives us a relation between κ and Φ_Δ:

$$\frac{\partial\kappa}{\partial Q_\Delta} = 8\pi \frac{\partial \Phi_\Delta}{\partial a_\Delta}. \tag{6.8}$$

Thus, if we know the surface gravity κ, then we can derive an expression for the potential Φ_Δ.

Finally, taking the variation of Eq. (6.3) and comparing it to Eq. (6.6) we arrive at the following equations:

$$a_\Delta \frac{\partial\beta}{\partial a_\Delta} + 8\pi r_\Delta Q_\Delta \frac{\partial\Phi_\Delta}{\partial a_\Delta} + 8\pi r_\Delta \frac{\partial V}{\partial a_\Delta} = 0, \tag{6.9}$$

$$\frac{r_\Delta}{2}\frac{\partial\beta}{\partial Q_\Delta} + Q_\Delta \frac{\partial\Phi_\Delta}{\partial Q_\Delta} + \frac{\partial V}{\partial Q_\Delta} = 0, \tag{6.10}$$

where $a_\Delta = 4\pi r_\Delta^2$ and, for convenience, we have defined $\beta := 2 r_\Delta \kappa$. Thus, given κ and Φ_Δ one can in principle, integrate Eqs. (6.9) and (6.10) to find V. Note that these equations are defined over the horizon parameters space $\mathcal{L}$, so they are completely *local* to the horizon Δ.

Recall that the general prescription for arriving at an explicit expression for surface gravity κ, for general isolated horizons, involves the fixing of the expansion μ as function of the horizon parameters. For this, one requires some input from the SSS solutions (where they exist). However, it is important to stress that the results of this section regarding a consistent Hamiltonian formulation and the validity of the first law are independent of the particular choice of normalization μ (and κ) that one makes. Thus, there is a consistent Hamiltonian for each choice. This is particularly important for those points of horizon parameter space where no SSS solutions exist and, therefore, no ''canonical'' normalization is available. Nevertheless, isolated horizons still exist, and are well defined for those points of parameter space. It is when we want to have a canonical choice of μ, and therefore of κ and M_Δ, that we are forced make contact with static solutions (for the allowed regions in $\mathcal{L}$).

In the remainder of this section, we focus our attention on static spherically symmetric solutions to the EYM equations described in Sec. II. We shall consider the three classes of SSS solutions described in Sec. II, and find expressions for their surface gravity κ and horizon masses M_Δ. As discussed before, a study of the three sectors of SSS solutions serves two purposes. First, it provides us with a way of fixing the normalization of l^a for *general* isolated horizons for those point of parameter space $\mathcal{L}$ where SSS solutions exist, and second, it will allow us to find, in the next Sec., new results regarding SSS solutions. This is because, even when the expressions we will find for κ and M_Δ are valid in the general framework, they are, in particular, also valid for SSS solutions. Some of the results of this and the next section, regarding SSS colored black holes, were already reported in [32].

Let us start with the electrically charged case, corresponding to the $P_\Delta = 0$ surface in $\mathcal{L}$. Since these solutions are nothing but electrically charged Reissner Nordström solutions, the expansion of n is given by [11]

$$\mu = -\frac{1}{r_\Delta} \tag{6.11}$$

and the surface gravity is given by

$$\kappa = \frac{1}{2 r_\Delta}\left(1 - \frac{Q_\Delta^2}{r_\Delta^2}\right). \tag{6.12}$$

We can now consider Eq. (6.8) and find that $\partial_{r_\Delta}\Phi_\Delta = -Q_\Delta / r_\Delta^2$. Then, asking Φ_Δ to vanish as $r_\Delta \mapsto \infty$, we have that

$$\Phi_\Delta = \frac{Q_\Delta}{r_\Delta}, \tag{6.13}$$

which corresponds precisely to the value of the electric potential on RN solutions. Equation (6.9) now implies that $\partial_{r_\Delta} V = 0$ and $\partial_{Q_\Delta} V = 0$. Since the restriction of V to the plane $P_\Delta = 0$ does not depend on P_Δ, the only possibility is that $V = \text{const}$ on the $P_\Delta = 0$ plane of $\mathcal{L}$. In order to fix the value of V we notice that the Schwarzschild one-parameter family of solutions—corresponding to zero electric field—is contained within the electric RN family. These solutions are purely gravitational since the gauge potential vanishes exactly, and it is known that for the pure Einstein theory one can set $V = 0$ (see [11]).

We now have expressions of the mass M_Δ, surface gravity κ, area a_Δ, and the electric potential Φ_Δ of any isolated horizon in terms of its fundamental parameters r_Δ, Q_Δ:

$$M_\Delta = \frac{r_\Delta}{2}\left[1 + \frac{Q_\Delta^2}{r_\Delta^2}\right]. \tag{6.14}$$

This is precisely the same form as in the Einstein-Maxwell theory. The total energy of the system E, related to on-shell value of the Hamiltonian, is given by

$$E = -H_t = M_{\text{ADM}} - M_\Delta, \tag{6.15}$$

which for the electric RN embedded family vanishes exactly.

Let us now consider embedded Abelian magnetic solutions. For this solutions, the electric charge Q_Δ vanishes, so the horizon mass variation formula, when restricted to the purely magnetic sector of the SSS space, takes the form

$$\delta M_\Delta = \frac{1}{8\pi}\kappa\, \delta a_\Delta. \tag{6.16}$$

The formula for the mass, Eq. (6.3), is given by

$$M_\Delta = \frac{1}{4\pi}\kappa a_\Delta + V(r_\Delta, Q_\Delta = 0, P_\Delta = 1). \tag{6.17}$$

The normalization factor μ remains the same as in the electrically charged case given by Eq. (6.11) and the surface gravity is given by

$$\kappa = \frac{1}{2 r_\Delta}\left(1 - \frac{P_\Delta^2}{r_\Delta^2}\right). \tag{6.18}$$

Then Eq. (6.8) implies that Φ_Δ is zero. The second set of equations (6.9) and (6.10) reduce to

$$\partial_{Q_\Delta} V = 0 \tag{6.19}$$

and

$$\partial_{r_\Delta} V = -\frac{r_\Delta}{2}\partial_{r_\Delta}\beta. \tag{6.20}$$

Using Eq. (6.18) we get the following equation that V should satisfy:

$$\partial_{r_\Delta} V = -\frac{P_\Delta^2}{r_\Delta^2}. \tag{6.21}$$

Now, recall that, for a given value of the magnetic charge P_Δ, one has black hole solutions for $r_\Delta \geqslant |P_\Delta|$ (the extreme case corresponding to $r_\Delta = |P_\Delta|$). In order to integrate Eq. (6.21), one has to choose some ''boundary conditions'' on the space $\mathcal{L}$. Our choice, motivated by consistency with the colored black holes (see below), is to set $V(r_\Delta = P_\Delta) = 0$. With this choice, V takes the form

$$V = \int_{|P_\Delta|}^{r_\Delta} \frac{P_\Delta^2}{\tilde{r}^2} d\tilde{r} = \frac{P_\Delta^2}{r_\Delta} - |P_\Delta|. \tag{6.22}$$

With this, the horizon mass M_Δ is given by

$$M_\Delta = \frac{r_\Delta}{2}\left(1 + \frac{P_\Delta^2}{r_\Delta^2}\right) - |P_\Delta| = M_{\rm ADM} - |P_\Delta|. \tag{6.23}$$

Let us now consider the case in which the Abelian solution has both electric and (unit) magnetic charge. The surface gravity is given by

$$\kappa = \frac{1}{2r_\Delta}\left[1 - \frac{(Q_\Delta^2 + P_\Delta^2)}{r_\Delta^2}\right]. \tag{6.24}$$

Equation (6.8) leads us to conclude that $\Phi_\Delta = Q_\Delta / r_\Delta$, and Eq. (6.9) takes the form

$$\partial_{r_\Delta} V = -\frac{(Q_\Delta^2 + P_\Delta^2)}{r_\Delta^2} + \frac{Q_\Delta^2}{r_\Delta^2} = -\frac{P_\Delta^2}{r_\Delta^2}. \tag{6.25}$$

Then, imposing again the boundary condition that V vanish on extremal magnetic solutions, we have that

$$V = \frac{P_\Delta^2}{r_\Delta} - P_\Delta. \tag{6.26}$$

The horizon mass is now

$$M_\Delta = M_{\rm ADM} - P_\Delta, \tag{6.27}$$

and the total energy is then

$$E = P_\Delta = 1.$$

Finally, there is the most interesting case, i.e., the family of colored black holes labeled by r_Δ and an integer n. Since these solutions correspond to the purely magnetic case, Eqs. (6.16), (6.17), (6.19), and (6.20) continue to hold. This last condition that the function $V = V(r_\Delta)$ should satisfy can be written as

$$V' = -\frac{r_\Delta}{2}\beta', \tag{6.28}$$

with the prime denoting differentiation with respect to r_Δ. (We are considering variation with fixed value of n.) Furthermore, by requiring that $M_\Delta \mapsto 0$ as $r_\Delta \mapsto 0$—coming from physical considerations—we arrive at the following relation:

$$M_\Delta = \frac{1}{2}\int_0^{r_\Delta} \beta(\tilde{r}) d\tilde{r}, \tag{6.29}$$

where the integration is again performed over the *space of parameters* of the n-colored black hole, labeled by the horizon radius r_Δ, and not over space-time. Let us note that for the $n=0$ solution, where β is known in closed form ($\beta = 1$), we arrive at $M_\Delta^{(n=0)} = r_\Delta/2 = \kappa a_\Delta/(4\pi)$, as expected.

Several remarks are in order. First, we must emphasize that the determination of V, and thus of M_Δ, relied on considerations involving only variations of quantities associated with the horizon Δ. Thus, the horizon mass is a well-defined quantity in the isolated horizons phase space $\mathcal{IH}$ (provided that a global normalization of μ exists). Second, the Hamiltonian horizon mass (HHM) defined by Eq. (6.3), when restricted to SSS configurations, does not agree with the usual definitions of mass that one finds in the literature (see, for instance, [17] and [33]). It should be stressed that Eq. (6.29) comes from a consistent Hamiltonian formulation, and is *not* a definition as occurs in other treatments.

As it was discussed at the end of Sec. V, the fact that SSS solutions to the EYM equations do not span the space $\mathcal{L}$ of horizon parameters is, in a sense, disturbing. It might seem that the Einstein-Yang-Mills system is in a different status than the Einstein-Maxwell-dilaton system where the space of horizon parameters is in a one-to-one correspondence with the space of static solutions. One would like to have a similar result in the EYM case. However, the SSS solutions span only a subspace of $\mathcal{L}$. Luckily, the spherically symmetric solutions in EYM theory do not exhaust all possible static solutions (as occurs in EMD theory); there are static solutions with axial symmetry that are not spherically symmetric [31]. With these results at hand, we propose a completeness conjecture in the next section.

VII. CANONICAL NORMALIZATION PROBLEM: A PROPOSAL

We have seen that in order for the isolated horizons scheme to define the surface gravity and the horizon mass of the colored black holes we needed to introduce a uniqueness conjecture $C1$ that guarantees that, given the isolated horizon parameters $a_\Delta, P_\Delta, Q_\Delta$, there would be at most one SSS solution. This was needed for, otherwise, the normalization of μ would not be uniquely specified given those parameters. The existing numerical evidence does indeed strongly support this conjecture. However, as we have mentioned before, and as is evident from the previous discussion, this is not sufficient in order to have the isolated horizon framework working for the EYM system to the same extent that it works, say, for the Einstein vacuum, Einstein-Maxwell, and

Einstein-Maxwell-dilaton systems. In order to achieve that, we would need to have a canonical normalization of μ for a ''complete'' set of values of the isolated horizon parameters. In the previously mentioned cases this canonical choice is given by the existence of static (and spherically symmetric) black hole solutions for all isolated horizon values of the parameters.[7]

There is strong numerical evidence against the validity of the analogous claim in the case of the EYM system. In fact in the regime of staticity and spherical symmetry there are, given a fixed value of a_Δ, only a discrete set of values of P_Δ for which there are black hole solutions. Moreover, within this regime there are no black hole solutions for any value of $P_\Delta \neq 1,0$ and $Q_\Delta \neq 0$. Thus, if we want to have any hope that any claim in that direction might be true, we must formulate it outside this restrictive regime. Indeed the fact that in EYM systems there are static black hole solutions that are not spherically symmetric (indicating that the theorem analogous to Israel's theorem is false in this case) already shows us that we must go beyond the SSS regime. In fact the solutions alluded to above are axially symmetric, instead of spherically symmetric, but seem to share, with the SSS solutions, the discreetness of the allowed values of P_Δ (at least to the extent that this issue has been studied [31]). Thus we have to go beyond this regime as well. In fact there are strong indications (see, for example, the discussion in [23]) that we must go beyond the static regime, and pose the conjecture in a broad enough setting that would still allow one to single out, for a given choice of IH parameters, a particular black hole solution and thus a canonical normalization of μ. This would be of course the class of stationary black hole solutions, where we would have to keep track also of the angular momentum, both at infinity J_∞ and at the horizon J_Δ. The completeness conjecture would thus be the following: $C2$: *For every value of the isolated horizon parameters* $a_\Delta, P_\Delta, Q_\Delta, J_\Delta$ *for which a space-time can be constructed, there exist also a stationary black hole solution with the same value of the parameters, now characterizing the Killing horizon.*[8]

Let us now consider some of the implications of this conjecture. First, a stationary black hole solution would be characterized by its parameters at infinity, $M_{\rm ADM}, P_\infty, Q_\infty, J_\infty$ and therefore the conjecture would imply the existence of a well-defined map Ψ: $(a_\Delta, P_\Delta, Q_\Delta, J_\Delta) \rightarrow (M_{\rm ADM}, P_\infty, Q_\infty, J_\infty)$ The failure of the no-hair conjecture would indicate that this map is not invertible. In fact we know that it would not be injective. Moreover, the map would be nontrivally four dimensional, in the sense that fixing, say, $J_\Delta = 0$ would not fix $J_\infty = 0$, as can be seen from the following expression [34]:

$$4\pi\Omega(J_\Delta - J_\infty) = \int_\Sigma d^3x [t^b \tilde{\mathbf{E}}^a_i \mathbf{F}^i_{ab} + \mathcal{L}_t(\mathbf{A}^i_a)\tilde{E}^a_i], \quad (7.1)$$

valid for stationary black hole solutions in EYM theory. Here, Σ is a maximal hypersurface intersecting the bifurcate horizon, and t^a is the projection to Σ of the time-translation Killing field. As it can be seen from Eq. (7.1), there is a bulk contribution to J_∞, the canonical angular momentum at infinity. Here, J_Δ is a particular definition of ''horizon angular momentum'' (given by a Komar integral), and Ω stands for the angular velocity of the horizon, i.e., the expression appearing in the first law,

$$\delta M_{\rm ADM} + \Phi_\infty \delta Q_\infty - \Omega\, \delta J_\infty = \frac{1}{8\pi}\kappa\, \delta a_\Delta, \quad (7.2)$$

where Φ_∞ is the ''electric potential'' at infinity. In fact the EYM system is, in this respect, rather different from the Einstein-Maxwell system, because in the latter one can disentangle, for example, the expression for $\Phi_\infty Q$ from the expression for ΩJ_∞, something that cannot be done in EYM [34], in which case the only relationship that can be obtained is given by the expression

$$8\pi(\Omega J_\infty - \Phi_\infty Q_\infty) = \int_\Sigma N(\pi_{ab}\pi^{ab} + 2\mathbf{E}^a_i \mathbf{E}^i_a)/h, \quad (7.3)$$

with π^{ab} the momentum conjugate to h_{ab}, the Riemannian metric on M, and N the ''lapse function.''

Assuming the validity of $C2$ one would have a canonical choice for the normalizations that could be used to uniquely define κ and M_Δ, the one provided by the Killing field of the stationary solution that is null at the horizon and that is normalized so that at infinity is a unit time translation. In order to make all these considerations more precise from the isolated horizon point of view, one needs to consider the extension of the formalism given in [25,35].

Now, let us concentrate for the moment in the SSS sector and see if we can understand the discreteness observed there in terms of this conjecture. In the analysis that led to the discovery of SSS black holes in EYM theory, one is fixing $P_\infty = 0$ because one is interested in solutions that are Abelian at large distances. Moreover, spherical symmetry evidently requires $J_\infty = 0$ and $J_\Delta = 0$. Moreover, the mixture alluded to before would prevent us from achieving this (spherical symmetry) unless we also set $Q_\infty = 0$. Thus we see that the (highly nonlinear) problem is given by four constraints in a four-dimensional space, and thus that the set of solutions is expected to be given by a discrete set (i.e., the linearized

[7] In fact, although this is true for the Einstein vacuum system, in the Einstein-Maxwell case we already have a potential problem, because if $Q > r_\Delta$, there is no such static solutions. The consistency of the whole scheme would require the impossibility of constructing a space-time containing an isolated horizon with such values of the parameters. This is a rather serious consequence of the present point of view, and one that should be testable. There is a strong correlation of these issues with the cosmic censorship conjecture that prevents us from violating the inequality $Q < r_\Delta$ for static solutions. It would seem that the IH formalism implies that one cannot construct initial data for a solution containing a black hole with values of the parameters that violate this inequality.

[8] In this statement, a space-time ''can be constructed'' whenever there exists an asymptotically flat solution to the EYM equation satisfying IH boundary conditions with the specified values of the horizon parameters.

problem about a given solution in the SSS sector has no nontrivial solution within the sector).

Next, let us consider some consequences of the completeness conjecture in the structure of the space $\mathcal{L}$. We have seen in Sec. VI that the space $\mathcal{IH}$ (where we now have to include distortion and rotation [25,35]) is foliated by ''symplectic leaves'' where the Hamiltonian formulation and the first law are valid. This foliation intersects the space $\mathcal{S}$ of stationary solutions and defines a one-parameter foliation of it. Now, if the $C2$ conjecture is valid, we have an isomorphism between $\mathcal{S}$ and $\mathcal{L}$, which then induces a canonical foliation of $\mathcal{L}$.

On the other hand, we must point out the following heuristic argument against the conjecture. Consider a stationary black hole solution in EYM theory; in order to be asymptotically flat the YM field strength must fall off rather rapidly at large distances. Thus the self-interaction of the fields must be falling off faster than the fields themselves and thus the fields must behave in the large distance limit as free fields, i.e., as Abelian fields. This suggests that the only possible values of the magnetic charge at ∞, P_∞ are the Abelian values $0,1,\ldots$ (there is, of course, no such restriction on the allowed value of the electric charge). This view is supported by the experience with the static spherically symmetric solutions. Thus, any stationary solution would be characterized at infinity by the parameters $M_{\rm ADM},Q_\infty,J_\infty$ (setting for the moment $P_\infty=0$ to simplify the discussion). On the other hand, the solution will be characterized by its horizon parameters, among them a_Δ. We know from the first law in its asymptotic infinity version, Eq. (7.2) (see the discussion in [23]), that the stationary black hole solutions are extremum of $M_{\rm ADM}$ at fixed $a_\Delta,Q_\infty,J_\infty$ within the constrained phase space (i.e., the space of allowed initial data, which as usual, can be identified with the space of solutions). Each such extremum is an isolated point in that space, so the manifold of stationary solutions is three dimensional. In [23] an argument is given that indicates that there would be a countable infinity of solutions for each value of the parameters $a_\Delta,Q_\infty,J_\infty$ which would generalize what happens in the case of the static solutions. This suggests that the manifold of stationary solutions is made of a countable infinity of connected three-dimensional components. On the other hand, the conjecture would seem to indicate that such a manifold must be four dimensional. The only way to avoid this would be to require the impossibility of constructing solutions of the equations representing asymptotically flat black hole space-times (not necessarily stationary) containing isolated horizons for all values of the isolated horizon parameters.

One would be tempted to take such position in view of the discretness of the SSS colored black holes with a fixed value of the horizon area a^0_Δ. Let $\{P^i_\Delta\}_{i=1}^\infty$ be the values of the horizon magnetic charges of the SSS colored black holes with horizon area a^0_Δ. One might want to argue that given a value of $P'_\Delta\in(0,1)$ that is not in that list, and is, say, between two of the values in the list, one cannot construct a solution representing asymptotically flat black hole space-time with an isolated horizon and, say $a_\Delta=a^0_\Delta,P_\Delta=P'_\Delta,Q_\Delta=0,J_\Delta=0$. Unfortunately this claim is evidently false: The recipe for constructing such a space-time is to give initial data that, upon evolution, would be static near the horizon and near infinity at least for a finite ''time'' interval. Take the equations for the SSS solution [Eqs. (10), (11), and (12) in [15]] and set $r_H=\sqrt{a^0_\Delta/4\pi}$, $w(r_H)=\sqrt{1-P'_\Delta}$, and evolve the elliptic equations up to, say, $r=2r_H$ (if we continue to evolve the equations attempting to obtain a static solution, we would find that w diverges, so the solution would not be asymptotically flat). For, say, $r\geqslant 5r_H$ take the initial data for the Schwarzschild solution. In the intermediate region $r\in(r_H,5r_H)$ take any interpolating function for w, and set the time derivatives of the functions w,m, etc. (i.e., the ''momenta'') to satisfy the constraints. The point is that the evolved space-time will be static in a neighborhood of the horizon, so, in particular, the horizon will be isolated, at least during some finite time interval (until the radiation coming from the intermediate region arrives at the horizon). Note that, generically, in this case the event horizon will fail to coincide with the isolated horizon.

The argument above suggests that the manifold of isolated horizon parameters $\mathcal{L}$ is indeed four-dimensional and thus the conjecture would require the identification of this four dimensional manifold with the infinite set of three-dimensional components that seem to constitute the manifold of stationary solutions. The best that can be hoped at this point is that $\mathcal{S}$ be dense in $\mathcal{L}$, a situation that would indicate that in the EYM theory there is much richer structure that what is found in, say, the Einstein-Maxwell system. In this case, the canonical normalization for l would be given by an appropiate limit (within $\mathcal{S}$) of solutions where the normalization exists (assuming, of course, that there is such limit).

On the other hand, the argument above is by not any means a tight proof, particularly so in the case of the conclusion about the Abelian nature of the allowed values of P_∞. As we have mentioned before, the validity of this conjecture, or some version of it (as, for example, a version based on the assumption that the manifold of stationary solutions is mapped into a dense subset of the isolated horizon parameters), seems to be the only reasonable way in which the isolated horizon scheme can be as successful in the general case as it has proved to be in the Einstein vacuum, Einstein-Maxwell, and Einstein-Maxwell-dilaton theories.

Needless is to say, further research is required in order to elucidate whether one of the scenarios considered above is correct or whether, in fact, the isolated horizon scheme fails to achieve in EYM theory the same degree of success that is attained in the previously treated cases.

VIII. SPHERICALLY SYMMETRIC STATIC SOLUTIONS: MASS AND HAIR

In this section we shall restrict our attention to the SSS sector of isolated horizons. One issue that has been considered for non-Abelian gauge theories is the relation that might exist between the existence of regular static, solitonic solutions, and ''hairy'' black hole solutions. This issue has been considered, for example, in [19] from heuristic and dimensional arguments. In this section, two main issues are studied. First, by restricting the Hamiltonian formulation for isolated black holes to the SSS sector, we can define the Hamiltonian horizon mass of SSS black holes in EYM theory. We then use this expression to show that this quasilocal definition together with some basic properties of Hamiltonian mechanics lead us to a formula relating HHM and ADM mass of the colored BH solutions with the ADM mass of the solitons of the theory. We also conclude that the positivity of the "total energy" spectrum of the colored black holes is related to their instability.

These results are quite surprising, because the IH formalism was developed to extend the notion of black holes to situations where radiation is present—and goes out to infinity—and one might have not expected to obtain new results already in the static sector of the theory.

As we have previously mentioned, the HHM M_Δ of a SSS BH does not correspond to any of the quasilocal definitions of the BH mass considered in the literature. Furthermore, M_Δ has the virtue of being constructed from a consistent Hamiltonian formulation which places it on a different status as the standard definitions.

To begin, let us calculate the value of the "total energy" E of the system for the three sectors of SSS solutions. In order to do this, we use a general argument from symplectic geometry that states that, within each connected component of the space of static solutions $\mathcal{S}$ embedded in the space of isolated horizons, the value of the Hamiltonian H_t remains constant [11]. Let us review this argument since it is essential for our discussion. The Hamilton equations of motion can be written as $\delta H=\Omega(\delta,X_H)$, where Ω is the symplectic form, δ is an arbitrary variation, and X_H is the Hamiltonian vector field. A static solution is one at which the Hamiltonian vector field either vanishes or generates pure gauge evolution. In either case, the symplectic structure evaluated on X_H and *any* arbitrary vector field δ vanishes. Therefore, for this point of the phase space, $\delta H=0$ for any direction δ. In particular $\delta H=0$ for variations relating two static solutions. Now, in the case of Einstein-Maxwell theory, the no-hair theorems ensure that all static solutions are given by the RN family. That is, the space of static solutions is, in that case, connected. Furthermore, since there is no energy scale in the theory, the only "preferred" value for H_t is zero [11].

What is the situation in Einstein-Yang-Mills theory? First, there is the Abelian family of electrically charged solutions, which represent a connected component, parametrized by M,Q. For these solutions, the basic reasoning of [11] applies and, as follows from Eqs. (6.14) and (6.15), one has to conclude that for these solutions $H=0$. However, there is a subtle modification in the case of magnetic Abelian solutions. These solutions represent a disconnected component labeled by one parameter, namely, the mass M (the magnetic charge P is fixed to be unity). As discussed in Sec. II, the EYM system possesses an energy scale given by the YM coupling constant, so, in principle, nonzero values of H are allowed. In the one-dimensional component corresponding to Abelian magnetic solutions, the value of E can be computed using Eqs. (6.15) and (6.23) and is given by $E=M_{\rm ADM}-M_\Delta=|P_\Delta|=1$. (We are taking the YM coupling constant $g_{\rm YM}=1$.)

Finally, let us consider colored black holes. Each connected component of the space of SSS colored black holes is one dimensional (parametrized by a_Δ), and solutions corresponding to distinct values of n belong to disconnected components. That is, the space SSS has a countable number of connected components. As we shall now show, for $n\geqslant 1$ the value of the Hamiltonian turns out to be *different* from zero: $H_t^n\neq 0$.

Recall that the general argument described above tells us that the (on-shell) value of the Hamiltonian is constant for each family labeled by n. This in particular implies that its value is independent of the radius r_Δ of the horizon. Thus one is allowed to take the limit

$$H^{(n)}=\lim_{r_\Delta\mapsto 0}[M^{(n)}_{\rm ADM}(r_\Delta)-M^{(n)}_\Delta(r_\Delta)]. \tag{8.1}$$

Now, it is known that the colored black holes converge pointwise to the Bartnik-McKinnon soliton solutions [18] and that the ADM mass satisfies $M^{(n)}_{\rm ADM}\mapsto M^{(n)}_{\rm BK}$ when $r_\Delta\mapsto 0$. Furthermore, the horizon mass of the black hole M_Δ goes to zero in this limit, so we can conclude that

$$H^{(n)}=M^{(n)}_{\rm BK}; \tag{8.2}$$

that is, the total value of the Hamiltonian equals the mass of the nth Bartnik-McKinnon (BK) soliton solution.

We now collect our results for colored black holes and arrive at the following unexpected relation:

$$M^{(n)}_{\rm ADM}(r_\Delta)=M^{(n)}_{\rm BK}+M^{(n)}_\Delta(r_\Delta). \tag{8.3}$$

Thus, we are in the position of writing an explicit formula for the ADM mass of the n colored black hole as function of r_Δ:

$$M^{(n)}_{\rm ADM}(r_\Delta)=M^{(n)}_{\rm BK}+\frac{1}{2}\int_0^{r_\Delta}\beta^{(n)}(\tilde{r})\,d\tilde{r}. \tag{8.4}$$

In Fig. 2, we show the values of the HHM as a function of the horizon radius r_Δ for the $n=1,2$ families. Note that for a given value of the horizon area, the higher n is, the lower the horizon mass of the corresponding black hole. In Fig. 3 the value of the horizon magnetic charge P_Δ is shown as function of the radius.

It is important to stress that, *a priori*, one would not expect to get the value of quantities defined at infinity, such as the difference of ADM masses in terms of purely local quantities at Δ. At this point one might raise the following objection to the construction of M_Δ for colored black holes: If we start by considering the equation $\delta M_\Delta=\kappa\,\delta a_\Delta/8\pi$ and try to integrate it along the one-dimensional curve defined for each n, one can "trivially" do so by using the usual form of the first law at infinity, which tells us that the general solution for M_Δ is given by $M_\Delta=M_{\rm ADM}+c$, with c a constant. Then one might argue that the "only" thing one is doing is to set c such that $M_\Delta(r_\Delta=0)=0$. Thus, one would conclude, the derivation is trivial and even the notion of a horizon mass would seem questionable. This argument would be perfectly valid had we *postulated* the equation $\delta M_\Delta=\kappa\,\delta a_\Delta/8\pi$. However, the nontrivial point here is that this equation

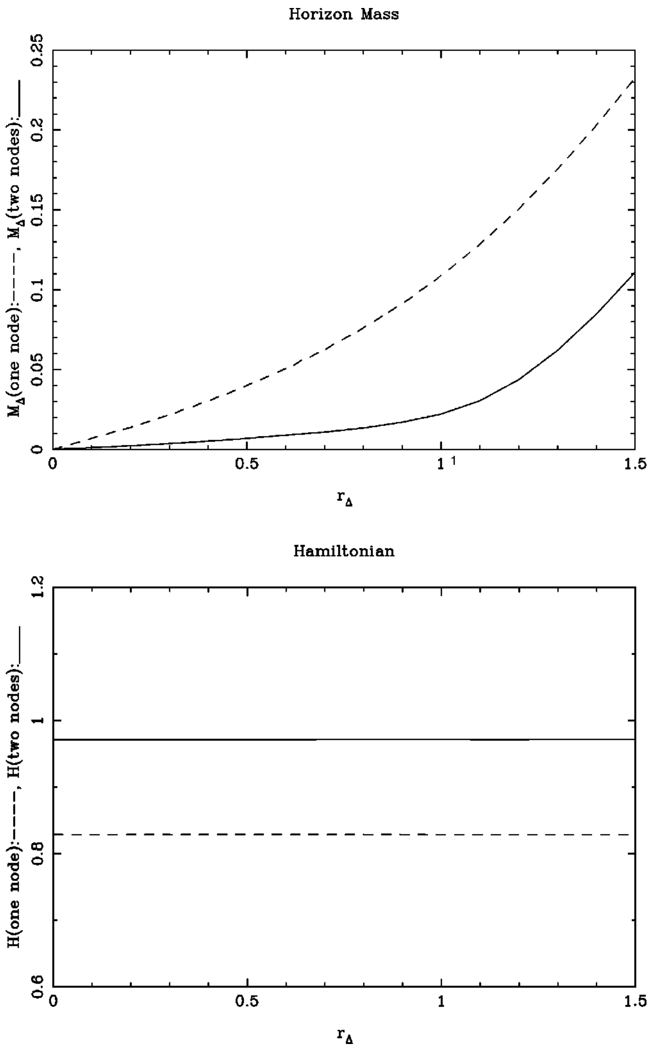

FIG. 2. The Hamiltonian horizon mass M_Δ and the total Hamiltonian H are shown as functions of the horizon radius r_Δ for the $n=1,2$ colored black holes.

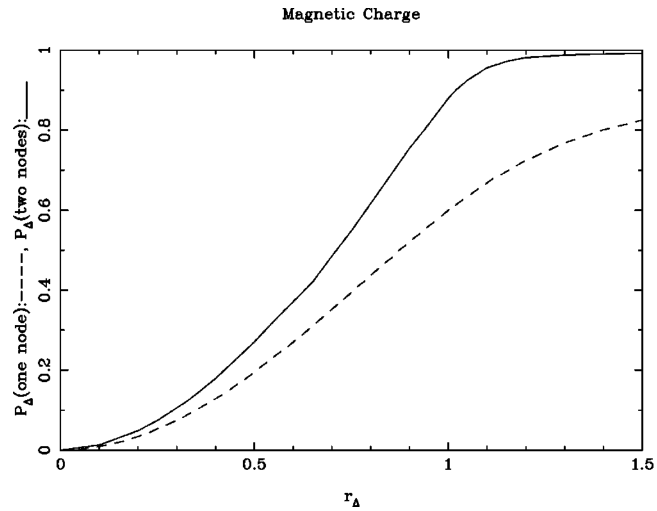

FIG. 3. The horizon magnetic charge P_Δ is shown as a function of the horizon radius r_Δ for the $n=1,2$ colored black holes.

comes as a consequence (and consistency requirement) from a Hamiltonian description respecting—physically motivated—boundary conditions. Therefore, even when the algebraic manipulations are simple, the final result is highly nontrivial.

We can now try to understand the physical meaning of the relation (8.4). Two facts are known about these solutions: first, we know that for fixed a_Δ these solutions represent saddle points of the ADM mass function M [23], and thus, as one can expect, for all values of n these solutions are unstable under small perturbations [36]. Let us now note that for the reported solutions in the literature (see, for instance, [18]), the BK mass is a monotonic function of n, starting at $M^1_{\rm BK}\approx 0.828$ and approaching 1 as n grows (in standard normalized units). The fact that the mass of the soliton, and therefore the total energy of the colored black holes, is positive confirms our expectation, coming from energetic considerations, that in general $M^{\rm ADM}\geqslant M_\Delta$. Indeed, since the difference between the horizon and ADM masses can be seen as the energy that is available for radiation to fall both into the black hole and to infinity, one can understand the nonzero value of the Hamiltonian as an indication that there is a potentiality for instability of the solution. In static solutions there is, of course, no radiation. Thus, to be precise, a positive value of the energy E means that, if we perturb slightly the initial data of a (unstable) static solution in such a way that the total energy is ''very close'' to E, then the resulting space-time will approach Schwarzschild in the future, and the total radiated energy to both infinity and the horizon will be equal to E. In conclusion, a necessary condition for the solution to be unstable is for the value of the total energy on the solution in question to be positive.

Let us conclude this section with four remarks.

(1) In the computation of the horizon mass for the magnetically charged Abelian solution, we required that the function V vanish for the extremal black holes. Let us now motivate this choice. First, it is known that for $r_\Delta\geqslant 1$ the n colored black holes approach the Reissner-Nordström magnetic solution (in a region around the horizon that expands unboundly with n) when $n\mapsto\infty$. Let us abuse notation for a moment and refer to these limiting solution as the ''$n=\infty$ colored black hole.'' From the numerical solutions reported in the literature one can notice that the horizon mass $M^{(n)}_\Delta = M^{(n)}_{\rm ADM} - M^{(n)}_{\rm BK}$ tends to zero when $n\mapsto\infty$ and $r_\Delta\mapsto 1$ from above. Now, if we require continuity from above for M_Δ on the space $\mathcal{L}$ of SSS solutions, we should require that

$$\lim_{r_\Delta\mapsto 1^+} M^{(\infty)}_\Delta = 0. \qquad (8.5)$$

But the horizon mass is given by $M^{(\infty)}_\Delta = \kappa a_\Delta/4\pi + V$ and $\kappa\mapsto 0$ as $r_\Delta\mapsto 1$ (since this corresponds to the extremal RN case). Thus, we conclude that $V\mapsto 0$ when $r_\Delta\mapsto 1$.

(2) Let us now return to the puzzle that we posed in Sec. II: how can we reconcile the treatments of static solutions at infinity and at the horizon? That is, when formulating the first law, say, at infinity, we only have two free parameters given by the ADM mass and electric charge Q_∞. However, at the horizon we have three parameters, the horizon area

a_Δ, the electric charge Q_Δ, and the magnetic charge P_Δ. Now, Eq. (6.6) tells us that their respective variations depend only on the horizon area and electric charge, which seems to be a contradiction. The proper setting of the problem again comes from the consistency of the Hamiltonian formulation. Recall that from Eq. (6.7) we concluded that the magnetic charge P_Δ is not free to vary at the horizon, but its variations are related to the variations of the horizon radius r_Δ. This means that the allowed values of P_Δ and r_Δ lie in submanifolds of codimension 1 embedded in the space $\mathcal{L}$. Thus, even when we can in principle have three independent parameters at the horizons, in practice when employing the Hamiltonian formulation, one is restricted to values of the magnetic charge that are determined by the horizon area (within each family labeled by n). Thus, the variations of the parameters both at the horizon and at infinity are consistent. This can also be seen in the following way. The results of different formalisms at infinity have shown that the ADM mass varies as [23,33]

$$\delta M_{\rm ADM}=\frac{1}{8\pi}\kappa\,\delta a_\Delta+\Phi_\infty\delta Q_\infty, \tag{8.6}$$

but since we know that, at a static solution, an arbitrary variation δ satisfies $\delta E=\delta(M_{\rm ADM}-M_\Delta)=0$, we have complete agreement with Eq. (6.6).

Now lets turn our view to the conflict that our completeness conjecture $C2$ seems to face in view of the fact that the colored black hole solutions, with different value of n, have different values of the Hamiltonian and the general argument, presented in Sec. VI, ensures that H must be constant over any submanifold of static (stationary) solutions. Thus, this general argument would lead us to conclude that H is constant over $\mathcal{S}$. The solution of the apparent paradox lies in the fact, already mentioned in Sec. VI, that the consistency of the Hamiltonian formulation leads to a foliation of phase space by symplectic leaves over which the Hamiltonian formulation is valid. These leaves intersect the manifold of stationary configurations, which is embedded in $\mathcal{I}H$. The intersection results in hypersurfaces of constant value of the Hamiltonian. In the case of SSS solutions each one of those corresponds to a single family labeled by a fixed value of n. If we now restrict our consideration to the hypersurface with $Q=0$, each of these families corresponds to a curve in the (r_Δ,P_Δ) plane. Namely, P_Δ becomes a function of r_Δ rather than an independent parameter. This again is similar to what happens in the case of a rotating body, where the manifold of stationary states does not correspond to a single value of the Hamiltonian, but the intersection of a symplectic leave with this manifold coincides with the curves of constant value of the Hamiltonian. Then, even when the space $\mathcal{L}$ if foliated by a ''continuum'' of leaves, the allowed Hamiltonian motions are restricted to lie within each of these level surfaces of the Hamiltonian H. This overall picture of the structure of the parameter space, is in fact, consistent with the situation in EMD. In this case, there is only one ''leave'' and, thus, the intersection with the manifold of static solutions has also only one leave; the value of H is indeed constant on the whole $\mathcal{S}$ (it is in fact zero).

It is intriguing to note that there seems to be a deep relation between the existence of nontrivial solitons and hairy black holes for which the charges are not independent (i.e., the possible SSS solutions are restricted to a discrete set of constant energy surfaces within $\mathcal{L}$). The fact that the consistency requirement on the Hamiltonian formulation leads to the ''nonstandard'' Hamiltonian framework for these cases [i.e., the foliation of phase space by the (symplectic) leaves on which there is a truly Hamiltonian framework] together with the constancy of the full Hamiltonian on stationary solutions can be regarded as explaining such relation. That is, on the intersection of each leave with the manifold $\mathcal{S}$ of stationary solutions, the full Hamiltonian is a constant. In the limit $a_\Delta\mapsto 0$ on each leave (provided such limit can be taken) we will find a soliton. The motion along the ''leave'' in $\mathcal{S}$ determines the mutual dependence of the isolated horizon parameters.

(3) It is now a general belief that the no-hair conjecture, even in its weakest form [8,19], is violated for some systems. One can thus hope that there be a uniqueness result for static solutions in terms of quasilocal parameters at the horizon. In Sec. VI we have put forward a ''horizon parameters uniqueness conjecture'' stating that all static BH solutions are characterized by their horizons parameters (''quasilocal charges'') in a unique way. In particular, it should be true that, given the horizon area a_Δ and the horizon electric charge Q_Δ and magnetic charge P_Δ, the static solutions be uniquely determined. For instance, if we set Q_Δ, given an arbitrary value for the magnetic charge $P_\Delta\in[0,1]$, there might be no solution, but if there is a solution, it should be unique. The numerical evidence available supports the conjecture (see Fig. 3).

(4) One other point already mentioned is the issue of the stability test provided by this type of analysis: It is only when $M_{\rm ADM}>M_\Delta$ in Eq. (6.15) that the solution can be unstable. One very clear example of this is given by the magnetic RN solution, which can be considered within both the Einstein-Maxwell theory and the EYM theory. This solution is stable within EM theory but unstable within EYM theory [37–39]. We can now understand this situation in the following way: In the former case the gauge connection cannot be globally described through a gauge field A_a as it corresponds to a nontrivial bundle. In this case one can nevertheless apply the IH formalism through the use of the duality symmetry of Maxwell theory. This results in the appearance of a term $\Phi_M P_\Delta$ taking the place of ΦQ_Δ in Eq. (6.3), and as is well known the evaluation of E as $M_{\rm ADM}-M_\Delta$ (in this case $H^{(0)}=0$ as there are no Abelian magnetically charged regular solitons) gives $E=0$, thus accounting for the stability of the solution. Let us look at what happens in EYM theory. In this case the solution can be described in terms of the gauge fields A_a^i because it is associated with a trivial bundle (with larger group) and there is therefore no term of the form $\Phi_M P_\Delta$ in Eq. (6.3). The P_Δ dependence of the mass M_Δ comes through V. As we have shown, the value of E for this solution is positive $(E=1)$, thus allowing for the instability of these solutions. Note that this same argument is valid also

for dyons with both electric and unit magnetic charge [see Eqs. (6.24)–(6.27)], and thus, it indicates a potential instability of these solutions.

Finally, let us conclude this remark by suggesting a ''rule of thumb'' for finding potentially unstable solutions, suggested by the EYM system. In the static family of solutions, consider the limit $r_\Delta \mapsto 0$. We have three possibilities: (i) We arrive at a regular solution with zero energy (i.e., Minkowski). This indicates that the whole family, labeled by r_Δ, is stable. (ii) There is a minimum allowed value of r_Δ corresponding to zero surface gravity. In this case, we can not conclude anything. (iii) In the limit one finds a regular solution with positive energy (a soliton different from the vacuum). In this case, the whole family of solutions (including the soliton) is potentially unstable. It would be interesting to reexamine, from this perspective, the (complete nonlinear) stability of the Einstein-Skyrme solitons and black holes [40].

IX. DISCUSSION

In this paper, we have studied the extension of the isolated horizon formalism to include the EYM system and found that it leads to a ''nonstandard'' Hamiltonian formulation. The main feature of this formulation is that it provides a foliation of phase space into symplectic leaves, in each of which we do get a standard Hamiltonian formulation. The framework nevertheless provides a powerful tool for studying some classical aspects of the theory already at the static level. In particular, we found an unexpected relation between the the ADM mass of a static spherically symmetric black hole solution, its horizon mass, and the ADM mass of the corresponding solitonic solution. These relationships were checked numerically in terms of the known numerical results obtained in the process of finding those solutions and thus the agreement can be seen as a check on the whole formalism.

An apparent tension and a challenge for the formalism are given by the existence of hairy solutions, where the number of charges at infinity and at the horizon do not coincide. We have been able to pinpoint the problem in a precise way using the isolated horizon formalism. This involves the analysis of the consistency of the Hamiltonian formulation and the nature of the first law. We have encountered difficulties in defining a canonical normalization for the vector l^a and, thus, for the horizon mass in general. We have proposed possible resolutions for this difficulty. Motivated by all these results and in order to have a satisfactory treatment of the EYM system within the framework, we have put forward a ''quasilocal uniqueness'' and a ''completeness'' conjectures for stationary black holes. In the case of the latter we have put forward arguments both in favor and against it, but the main point is that some version of it seems to be the only possibility to have the isolated horizon framework working in EYM theory to the same extent that it does in, say, Einstein-Maxwell theory.

The present work can be generalized in several directions. First, our analysis allows us to propose an isolated horizon treatment for general theories containing nontrivial black hole and soliton solutions; it should be possible to apply the type of analysis presented here to these theories where nontrivial regular static solutions have been found. In particular, Einstein-Yang-Mills-Higgs, Einstein-Yang-Mills-dilaton, and Einstein-Skyrme theories are examples in which there are both solitonic and black hole solutions. In all these cases, formulas analogous to the EYM case can in principle be found by a straightforward application of the analysis carried out in the last section. In particular, one should be able to to compute the ''total energy'' of the ''hairy'' black holes to test for a potential instability.

Second, one can use the very recent results of Ashtekar and collaborators who have been able to extend the isolated horizons framework to include *distorted* horizons [25] as well as *rotating* horizons [35]. Thus, one should be able to use the formalism for distorted horizons in order to study colored black holes which are static but not spherically symmetric [31]. This analysis would be a first check for the ''quasilocal uniqueness conjecture $C1$'' that we have proposed [41]. The discussion of the preceding sections suggests that by restricting our attention to nondistorted, nonrotating horizons we were led to a consistent but ''incomplete'' formalism in the EYM system; a complete treatment (i.e., one providing a canonical choice of ''normalization'' for all $\mathcal{L}$, based on stationary solutions which are contained in the formalism) of EYM isolated horizons should be given within the context of distorted and rotating isolated horizons [25,35]. It would be interesting to investigate this matter once Refs. [25,35] are made public.

Finally, the present analysis shows that the so-called colored black hole solutions provide a nontrivial testing ground for the approach of [1] to evaluate the statistical mechanical entropy of black holes. In particular it is known that by selecting the value $\gamma = \ln 2/(\pi\sqrt{3})$ for the Immirzi parameter γ (which amounts to selecting one among a continuous choice of unitarily inequivalent quantum theories corresponding to the same classical theory), the standard result $S = A/4l_P^2$ is obtained for the Einstein vacuum and Einstein-Maxwell cases. It is worth pointing out that this choice can be made only once and that it is conceivable that say that the choice needed in the case of Einstein-Maxwell black holes might have been different than the choice needed for pure gravity black holes. Now, it would be of interest to check whether these results are also valid when non-Abelian gauge fields are present.

ACKNOWLEDGMENTS

We would like to thank A. Ashtekar, C. Beetle, S. Fairhurst, and R. Wald for discussions and correspondence. We are also grateful to the Center for Gravitational Physics and Geometry for its hospitality. This work was in part supported by DGAPA-UNAM Grant No. IN121298, by CONACyT Grants Nos. J32754-E and 32272-E, by a NSF-CONACyT collaborative grant, by NSF Grants Nos. INT9722514 and PHY95-14240, and by the Eberly research funds of Penn State.

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